In Class Examples

Finite Subsets

We begin by noting that any finite subset must have some maximum element. Thus, we can enumerate all the finite subsets by enumerating those with largest element 0, followed by largest element 1, followed by largest element 2, etc. Each step of this has only a finite number of elements, and every finite subset will eventually be output.

A Strange Set

(a) In the first step, we remove an interval of length $1/3$. For the second, we remove two intervals of size $1/9$, for a total of $1/3 \times 2/3$. For the third, we remove 4 intervals of size $1/27$, for a total of $1/3 \times (2/3)^2$. Therefore, the total amount removed is $\sum_{i=0}^{\infty} \frac{1}{3} \left(\frac{2}{3}\right)^i = \frac{1}{3 - 2/3} = 1$

(b) Consider the ternary representation of a number $r \in [0,1]$. By excluding the interval $(1/3, 2/3)$, we remove all numbers whose first digit is a 1. By excluding the remaining middle thirds, we remove all numbers whose second digit is a 1. Continuing in this fashion, we see that the Cantor Set consists of all those numbers whose ternary representations do not include a 1. But this is in bijection with all the binary numbers (just divide each digit by 2), and thus is in bijection with the entirety of $[0,1]$

Group Problems

Infinites

(a) The function $\frac{1}{1-x}$ almost works, but it misses the interval $[0,1)$. To fix this, we just do $\frac{1}{1-x} - 1$.

(b) We want to map the numbers $[0,1/2)$ to $[0,\infty)$, which we can do tweaking part (a) and getting $\frac{1}{1-2x} - 1$ and then we map $[1/2,1)$ to $(-\infty,0)$ by doing the same thing (specifically $-(\frac{1}{1-2x} - 1))$. This hits 0 twice, but that’s actually OK since we only need to show it’s at least as big as $\mathbb{R}$.

The Hotel Infinity

- Just have everybody move to the room $k$ greater than theirs and insert the new people in rooms 1,2,...,$k$

- Have everybody double their room number and then put the new people in all the odd numbered rooms.

- Assign a pair $(n,m)$ to each of the new people, where $n$ is their bus number and $m$ is their index within that bus. From lecture, we know the set of all pairs is countable and so we can assign a number $k_{mn}$ to each such pair. After doing this, we really only have infinitely many more people, so just use part (b).

The Limits of Induction

- Base case: 0 copies has size 0 and thus is finite. Suppose that $\mathbb{Z}_2^n$ is finite. Then $|\mathbb{Z}_2^{n+1}| = 2|\mathbb{Z}_2^n|$, so is finite by the inductive hypothesis.

- These are just all the strings of 0s and 1s of infinite length, so we can interpret them as all the binary numbers in the range $[0,1]$ and they are thus uncountable.