Lecture Review

Lecture 21: Probability Basics

- Understand the definitions of sample space, event, and probability distribution.
- Uniform Distribution: randomly picking one of \( n \) different balls from a bucket, each with prob \( 1/n \).
- Binomial (Bernoulli) Distribution: the number of heads from \( n \) independent coin flips with bias \( p \).
- Understand what it means for two events \( A \) and \( B \) to be independent. Formally, \( P(A|B) = P(A) \).

Lecture 22: Random Variables and Great Expectations

- A random variable (RV) is a function that maps every element in the sample space to a value.
- Intuitively, the expected value of a random variable is a weighted average over all elements in the sample space. Formally, \( E[X] = \sum_{A \in \Omega} X(A) \cdot P(A) \).
- An indicator RV for an event \( A \) is a RV that has value 1 for elements in \( A \), and 0 otherwise.
- Linearity of Expectation says that \( E[X + Y] = E[X] + E[Y] \). This applies even if \( X \) and \( Y \) are not independent. This can be very useful in conjunction with indicator RVs.

Independence and Conditional Probability

Consider independently flipping 3 fair coins.
Let \( A \) be the event that there are exactly 2 heads.
Let \( B \) be the event that the first and last coin land on opposite sides.
Question: Are \( A \) and \( B \) independent?

Inclusion-Exclusion in Probability

For any events \( A \) and \( B \), \( P(A \cup B) = P(A) + P(B) - P(A \cap B) \).
Suppose you are randomly dealt a 5-card poker hand.
Let \( A \) be the event that there is at least a pair of some rank.
Let \( B \) be the event that you are dealt at least one Jack.
Question: What is \( P(A \cup B) \)?

Linearity of Expectation

Consider independently flipping \( n \) fair coins, and lining them up on the table. What is the expected number of strings of 3 consecutive heads?
Group Problems

Please work on these problems in groups of 3 people. When a problem is solved, make sure everybody in your group understands the solution. Be prepared to present your solution to the class.

Understanding Independence

Question: Are the events $A$ and $B$ in the above picture independent?

What About Three-Ways?

Suppose events $A, B, C$ are pairwise independent. Is it necessarily true that $P(A) = P(A | B \cap C)$? If true, prove it. If false, provide a counterexample.

Hand-Raising

Consider the $n$ students in this recitation section. Suppose that for any given question asked by the TA, each student knows the answer with $2/3$ probability, but given he knows the answer, only volunteers with probability $1/2$. If the TA asks a question, what is the expected number of students that will know the answer and volunteer a solution?