Lecture Review

Lecture 13: Recurrences and Continued Fractions

- **The GCD Algorithm**: You should know how the Euclidean algorithm works.
- **Continued Fractions**: Know what a continued fraction is. Numbers with finite continued fraction expansions are exactly the rationals. For example, \( \frac{10}{7} = 1 + \frac{1}{2 + \frac{1}{3}} \), which we can also write as \([1, 2, 3]\).

Lecture 14: Generating Functions II

- **Solving Recurrences**: Use the recurrence to get an equation for the generating function, which we can solve; then use the geometric series formula to get a closed form for the coefficients.
- **Catalan Numbers** come up everywhere; it’s good to know the recurrence, to be able to derive the generating function, and to know the closed form.
- **Manipulating Generating Functions**: Adding or subtracting two generating functions adds or subtracts the corresponding series; multiplying two generating functions convolves the corresponding series.

GCD

The main idea behind the Euclidean algorithm is that \( \text{gcd}(a, b) = \text{gcd}(b \mod a, a) \). Why does this work?

Generating Functions for Recurrences

Consider the recurrence

\[
\begin{align*}
G(0) &= 2 \\
G(1) &= 5 \\
G(n) &= 5G(n-1) - 6G(n-2), \quad n \geq 2,
\end{align*}
\]

the first few terms of which are 2, 5, 13, 35, . . .

Using generating functions, find a closed form for \( G(n) \).

Manipulating Generating Functions

If we know that \( f(x) \) is the generating function for \( F(n) \) and \( g(x) \) is the generating function for \( G(n) \), we can combine the two to get generating functions for many other sequences. In particular:
We’re somewhat used to recurrences that refer back to a fixed number of previous terms; how can we deal with recurrences that aren’t so simple? For example, if \( F(0) = 1 \) and \( F(n) = \sum_{k=0}^{n-1} F(k) \), how could we find a closed form for \( F(n) \)?

### Group Problems

Please work on these problems in groups of 3 people. When a problem is solved, make sure everybody in your group understands the solution. Be prepared to present your solution to the class.

#### Vandermonde Revisited

You gave a counting argument in a homework to show that \( \sum_{k=0}^{n} \binom{r}{k} \binom{s}{n-k} = \binom{r+s}{n} \). Prove this fact again, this time with generating functions.

#### Summing Binomial Coefficients

Using generating functions, find a closed form for the sum \( \sum_{k=0}^{i} (-1)^k \binom{n}{k} \) in terms of \( i \) and \( n \).

*Correction:* this was supposed to be \( (-1)^{i-k} \) instead of \( (-1)^k \). As it was originally stated, the problem is harder — but still possible, given the tools you have.

#### Order of Operations

Say we have a function of 2 variables \( f(x, y) \), and we have \( n \) variables \( (a_1, a_2, \ldots, a_n) \). How many ways can we combine these variables using \( f \), in order, to get a single value? For example, with three variables, we can do \( f(f(a_1, a_2), a_3) \) or \( f(a_1, f(a_2, a_3)) \).

<table>
<thead>
<tr>
<th>Generating Function</th>
<th>Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) + g(x) )</td>
<td>( F(n) + G(n) )</td>
</tr>
<tr>
<td>( f(x)g(x) )</td>
<td>( \sum_{k=0}^{n} F(k)G(n-k) )</td>
</tr>
<tr>
<td>( f(x)/(1-x) )</td>
<td>( \sum_{k=0}^{n} F(k) )</td>
</tr>
<tr>
<td>( cf(x) )</td>
<td>( cF(n) )</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>( (n+1)F(n+1) )</td>
</tr>
<tr>
<td>( xf(x) )</td>
<td>( F(n-1) )</td>
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