GCD

We know that $\gcd(a, b) = \gcd(a, b-a)$ — if any number divides both $a$ and $b$, it must divide the difference, and any number dividing both $b-a$ and $a$ divides $a$ and $b$ (as you saw in 21-127). Repeatedly subtracting, we find that $\gcd(a, b) = \gcd(a, b \mod a) = \gcd(b \mod a, a)$.

Generating Functions for Recurrences

Using the recurrence, we can get the following equation for the generating function: $g(x) = 5xg(x) - 6x^2g(x) + 2 - 5x$. We can solve this to get $g(x) = \frac{2-5x}{1-3x+6x^2}$. Using partial fractions, we find that the quotient is equal to $\frac{1}{1-2x} + \frac{1}{1-3x}$. But the left-hand term is the generating function for $2^n$ and the right-hand term is the generating function for $3^n$; therefore, $G(n) = 2^n + 3^n$.

Manipulating Generating Functions

If $f(x)$ is the generating function, we know that $\sum_{k=0}^{n} F(k)$ has $f(x)/(1-x)$ as its generating function. To shift this to $\sum_{k=0}^{n-1} F(k)$, all we need to do is multiply by $x$, giving $xf(x)/(1-x)$. This almost gives us back $f(x)$: the only thing that goes wrong is the constant term, which ends up as 0 — but we can fix that by adding 1.

We end up getting the equation $f(x) = xf(x)/(1-x) + 1$, which we can solve to get $f(x) = (1-x)/(1-2x) = 1/(1-2x) - x/(1-2x)$. We know that $1/(1-2x)$ is the generating function for $2^n$, and $x/(1-2x)$ is the same generating function shifted by one (ie. $2^n-1$ for $n \geq 1$, and 0 when $n = 0$). We end up with $F(n) = 2^{n-1}$ if $n \geq 1$, and $F(0) = 1$.

Group Problems

Vandermonde Revisited

The generating function for $\binom{r}{k}$ (as a function of $k$) is $(1+x)^r$; the generating function for $\binom{s}{k}$ is $(1+x)^s$. The sum on the left is just the convolution of these, so its generating function is $(1+x)^r(1+x)^s = (1+x)^{r+s}$, which is the generating function for $\binom{r+s}{n}$ (as a function of $n$).

Summing Binomial Coefficients

The generating function for $(-1)^k$ is $\frac{1}{1+x}$ (which is easier to see if you think of it as $\frac{1}{1-(-x)}$). The generating function for $\binom{n}{i}$ (as a function of $k$) is $(1+x)^n$. Convolving these gives us the generating function $\frac{1+x}{1+x} = (1+x)^n$, which is the generating function for $\binom{n-1}{i}$ (as a function of $i$).

Order of Operations

We can turn these expressions into binary trees, recursively. A variable is a leaf, and to turn $f(L, R)$ into a tree, we can convert $L$ into a tree and $R$ into a tree, and combine the results as the left and right subtree of a new tree.

As we’ve seen, the Catalan numbers count the number of binary trees with $n$ leaves, so they also count how many ways we can combine the $n$ variables using $f$. 