Lecture Review

Lecture 4: Counting I: Choice Trees and Correspondences

**Correspondence principle.** If the elements of two (finite) sets can be placed into 1-1 correspondence, then the two sets have the same size.

**Choice trees.** Say we want to figure out the size of some set $S$. Suppose that we can single out any particular $x \in S$ by first making one of $P_1$ choices, then one of $P_2$ choices, ..., then one of $P_k$ choices. Then the product rule says that

$$|S| = P_1 P_2 \ldots P_k.$$

Two important conditions must be met in order for the product rule to work.

- Every $x$ must be reached by some sequence of choices.
- Each sequence of choices must lead to a distinct $x$ (otherwise you may be over-counting)!

**Binomial coefficient.** Know the formula for binomial coefficients: $inom{n}{k} = \frac{n!}{(n-k)!k!}$, and know how/when to use it.

Lecture 5: Counting II

**Multinomial coefficients:** The number of ways to arrange $n$ symbols with $r_1$ symbols of type 1, $r_2$ symbols of type 2, ..., $r_k$ symbols of type $k$ is

$$\frac{n!}{r_1! r_2! \ldots r_k!}.$$

**Pirates and gold:** Understand why the number of ways to distribute $k$ bars of gold among $n$ pirates is $\binom{n+k-1}{n-1}$. The same expression gives you

- the number of non-negative integer solutions to the equation $x_1 + x_2 + \ldots + x_n = k$, and
- the number of multisets of size $k$ formed using $n$ distinct elements.

Counting Two Ways

- Use a counting argument to show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

- Let $n$ be a positive integer and let $S = \{1, 2, 3, \ldots, n+1\}$. Consider the set

$$T = \{(x, y, z) \in S^3 : x < z \text{ and } y < z\}$$

- Show that $|T| = \sum_{k=1}^{n} k^2$.
- Show that $|T| = \binom{n+1}{2} + 2 \binom{n+1}{3} = \frac{n(n+1)(2n+1)}{6}$.
- Conclude that $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ (dot proof from homework 2).
Group Problems

Please work on the following problems in groups of 3 people. When a problem is solved, make sure everybody in your group understands it, and then tell the TA.

Counting Functions

Let \( A \) be a set with \( n \) elements, and let \( B \) be a set with \( m \) elements.

(a) How many different functions are there from \( A \) to \( B \)? Recall that a function \( f \) must map every element of \( A \) to some element of \( B \).

(b) Assume \( n \leq m \). How many different injective (1-to-1) functions are there from \( A \) to \( B \)? Recall that a function is an injection if every element in \( B \) is hit at most once.

(c) Assume \( n = m \). How many different bijective functions are there from \( A \) to \( B \)? Recall that a function is a bijection if each element in \( B \) is hit exactly once.

Manhattan Paths

A rook has been placed in the lower left hand corner of an \( n \) by \( n \) chess board. You must move the rook to the upper right hand corner of the board using only legal moves (the rook is only allowed to move in straight lines), and the sequence of moves must cover exactly \( 2n - 1 \) squares (this is the minimum possible number of squares). How many possible paths are there?