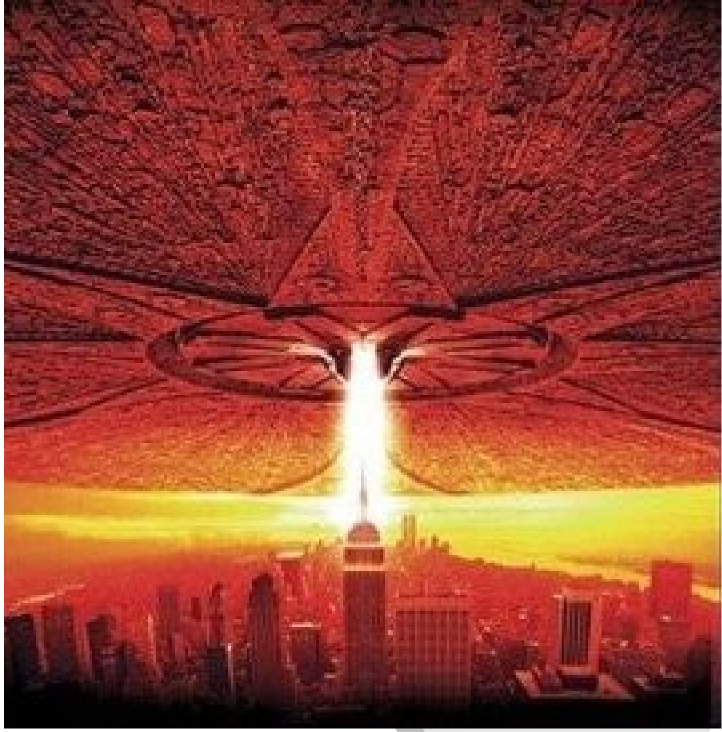
Bayes Nets II: Independence Day



CS 5300 / CS 6300
Artificial Intelligence
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Many slides courtesy of Dan Klein, Stuart Russell, or Andrew Moore

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www.cs.utah.edu/~hal/courses/2010S_AI

Reasoning Patterns and D-Separation

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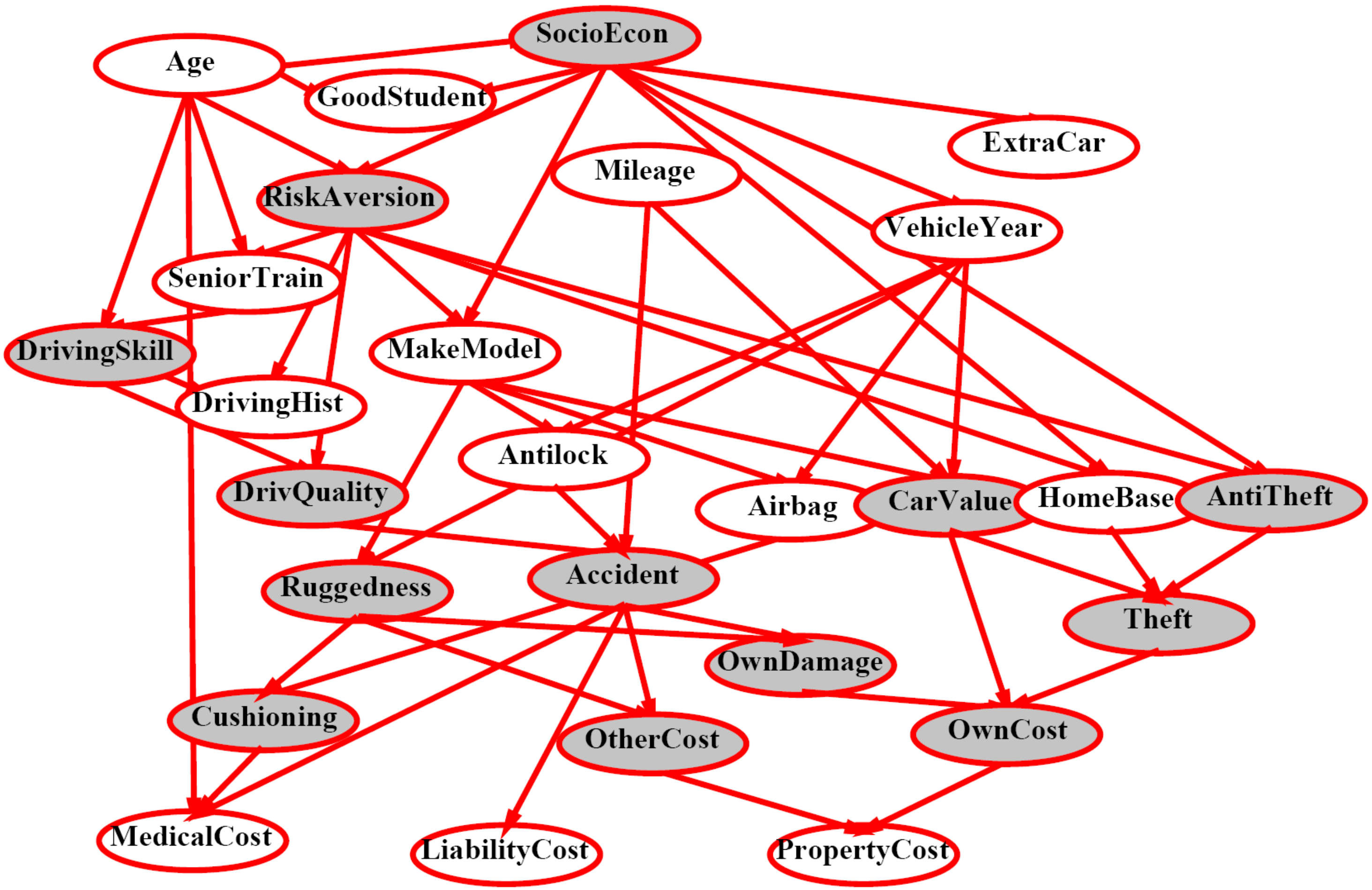
Example Bayes' Net



Bayes' Nets

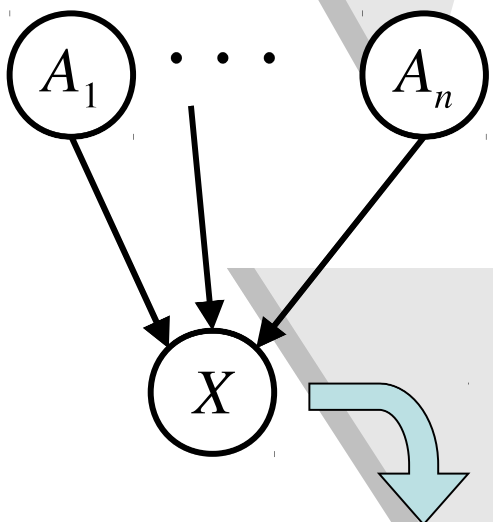
- A Bayes' net is an efficient encoding of a probabilistic model of a domain

- Questions we can ask:
  - Inference: given a fixed BN, what is P(X | e)?
  - Representation: given a fixed BN, what kinds of distributions can it encode?
  - Modeling: what BN is most appropriate for a given domain?

Bayes' Net Semantics

- A Bayes' net:
  - A set of nodes, one per variable X
  - A directed, acyclic graph
  - A conditional distribution of each variable conditioned on its parents (the *parameters* θ)

$$P(X|a_1 \dots a_n)$$



$$P(X|A_1 \dots A_n)$$

- Semantics:
  - A BN defines a joint probability distribution over its variables:

$$P(x_1, x_2, \dots x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i))$$

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Building the (Entire) Joint

- We can take a Bayes' net and build any entry from the full joint distribution it encodes

$$P(x_1, x_2, \ldots x_n) = \prod_{i=1}^{n} P(x_i | \mathit{parents}(X_i))$$

  - Typically, there's no reason to build ALL of it
  - We build what we need on the fly

- To emphasize: every BN over a domain implicitly defines a joint distribution over that domain, specified by local probabilities and graph structure

Example: Coin Flips

$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$\cdots$

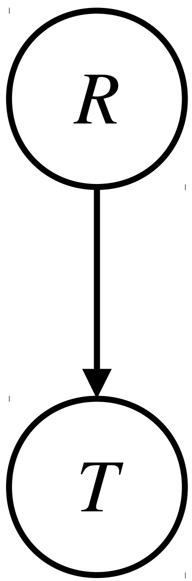
$P(X_n)$

h	0.5
t	0.5

$$P(h, h, t, h) =$$

Only distributions whose variables are absolutely independent can be represented by a Bayes' net with no arcs.

Example: Traffic



$P(R)$

r	1/4
¬r	3/4

$P(T|R)$

r →

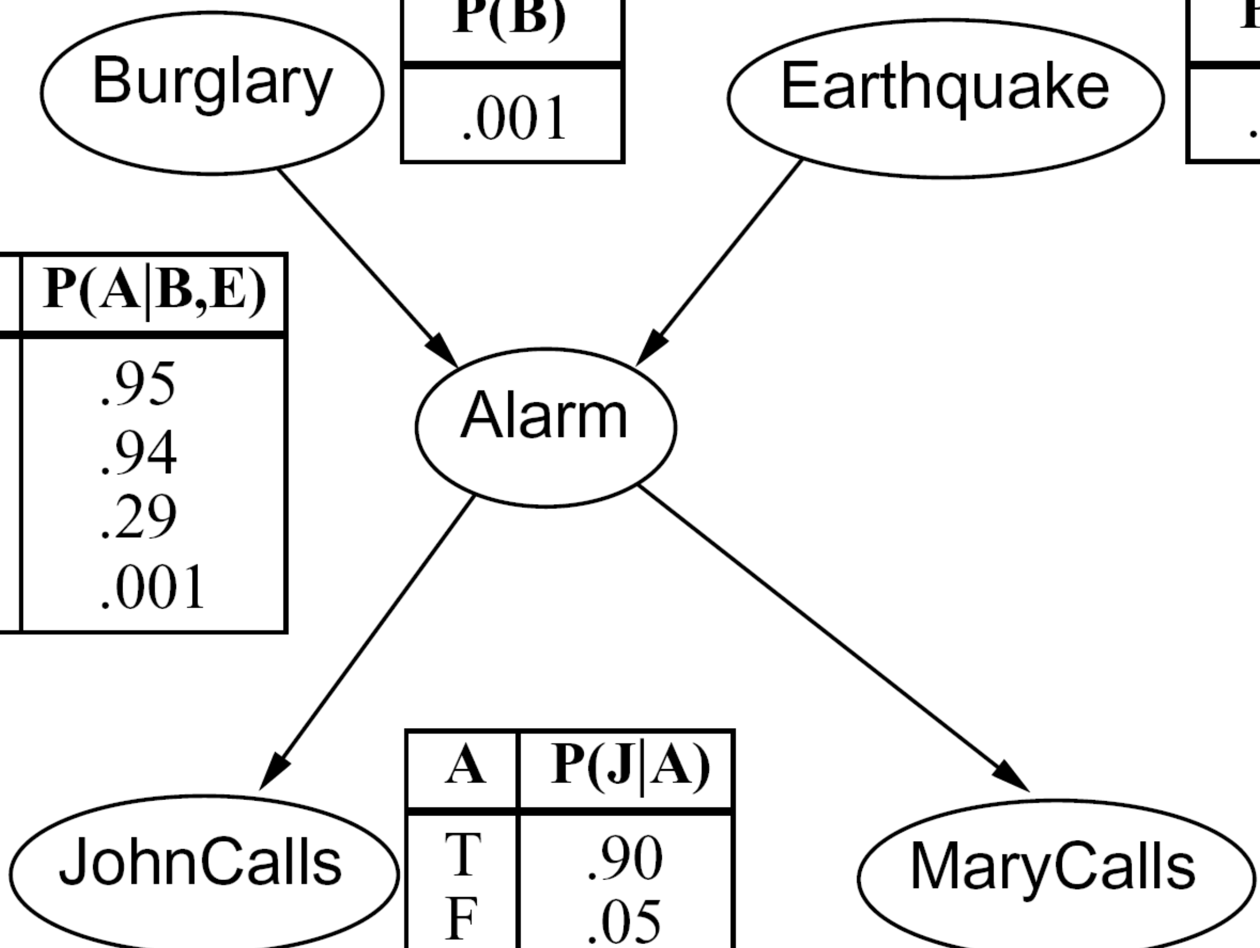
t	3/4
¬t	1/4

¬r →

t	1/2
¬t	1/2

$$P(r, \neg t) =$$

Example: Alarm Network



P(B)
.001

P(E)
.002

B	E	P(A|B,E)
T	T	.95
T	F	.94
F	T	.29
F	F	.001

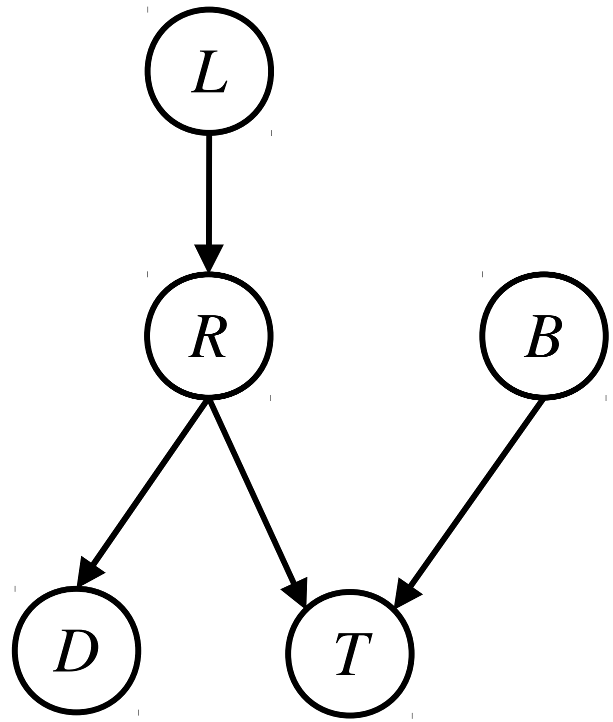
A	P(J|A)
T	.90
F	.05

A	P(M|A)
T	.70
F	.01

$$P(b, e, \neg a, j, m) =$$

Example: Traffic II

- Variables
  - T: Traffic
  - R: It rains
  - L: Low pressure
  - D: Roof drips
  - B: Ballgame



Size of a Bayes' Net

- How big is a joint distribution over N Boolean variables?
- How big is an N-node net if nodes have k parents?
- Both give you the power to calculate
- BNs: Huge space savings!
- Also easier to elicit local CPTs
- Also turns out to be faster to answer queries (next class)

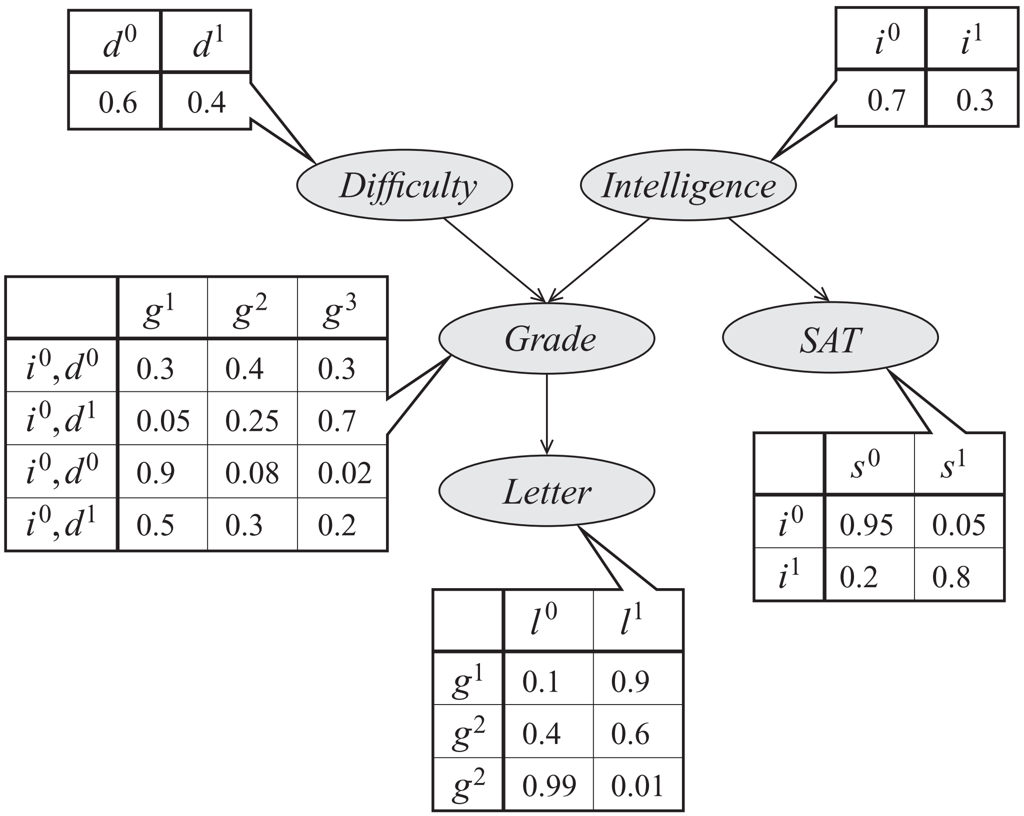
$$P(X_1, X_2, \ldots X_n)$$

Bayes' Nets

- So far:
  - What is a Bayes' net?
  - What joint distribution does it encode?
- Next: how to answer queries about that distribution
  - Key idea: conditional independence
  - Last class: assembled BNs using an intuitive notion of conditional independence as causality
  - Today: formalize these ideas
  - Main goal: answer queries about conditional independence and influence
- After that: how to answer numerical queries (inference)

Bayesian Network: Student Model

Graph and CPDs



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0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0,d^0	0.3	0.4	0.3
i^0,d^1	0.05	0.25	0.7
i^0,d^0	0.9	0.08	0.02
i^0,d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

Val(I)={i^0=low intelligence, i^1= high intelligence}

Val(D)={d^0=easy, d^1=hard}

Val(G)={g^1=A, g^2=B, g^3=C}

Val(S)={s^0=low, s^1=high)

Val(L)={l^0=weak, l^1=strong}

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

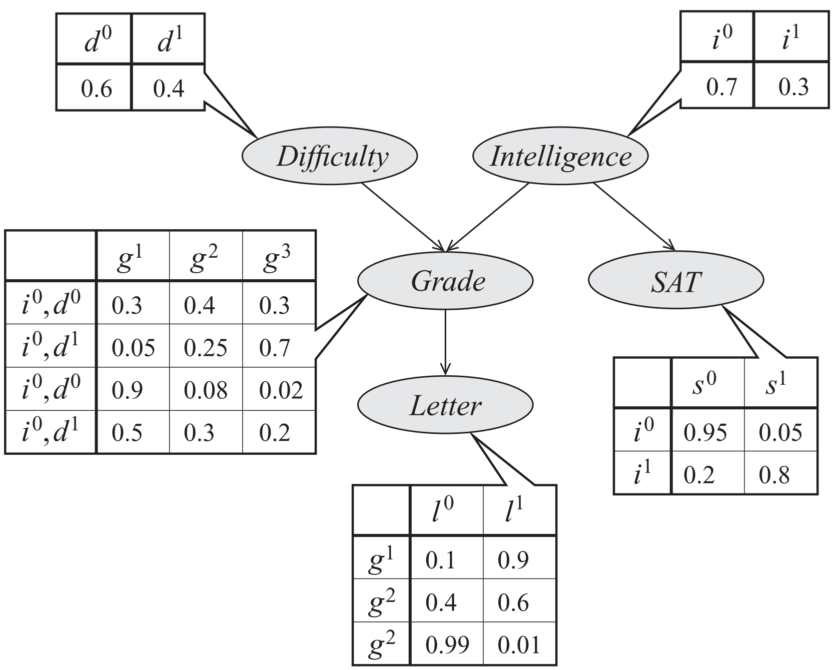
$$P(i^1,d^0,g^2,s^1,l^0) = P(i^1)P(d^0)P(g^2 \mid i^1,d^0)P(s^1 \mid i^1)P(l^0 \mid g^2)$$

$$=0.3 \cdot 0.6 \cdot 0.08 \cdot 0.8 \cdot 0.4=0.004608$$

Chain rule for Bayesian network

Reasoning Patterns

Reasoning about a student George using the model



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- Causal Reasoning
  - George is interested in knowing as to how likely he is to get a strong letter (based on intelligence, difficulty)?
- Evidential Reasoning
  - Recruiter is interested in knowing whether George is intelligent (based on letter, SAT)

Causal Reasoning

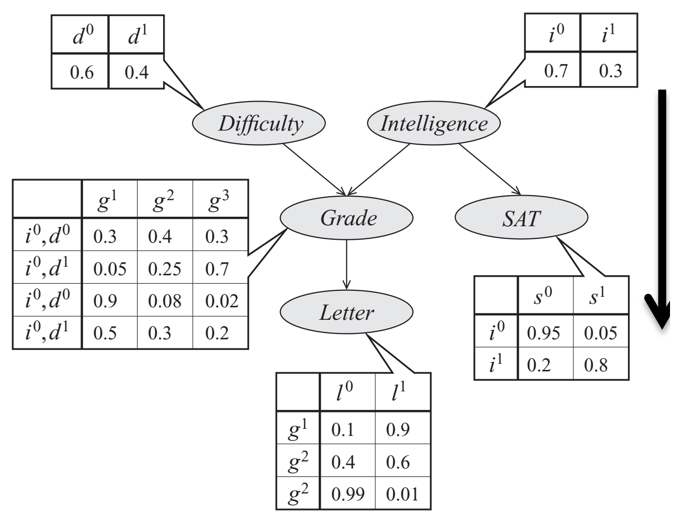
1. How likely is George to get a strong letter (knowing nothing else)?
- $P(l^1)=0.502$
- Obtained by summing-out other variables in joint distribution

2 But George is not so intelligent (i^0)
- $P(l^1|i^0)=0.389$

3. Next we find out ECON101 is easy (d^0)
- $P(l^1|i^0, d^0)=0.513$

Observe how probabilities change as evidence is obtained

$$P(D,I,G,S,l^1) = \sum_{D,I,G,S} P(D)P(I)P(G \mid D,I)P(S \mid I)P(l^1 \mid G)$$



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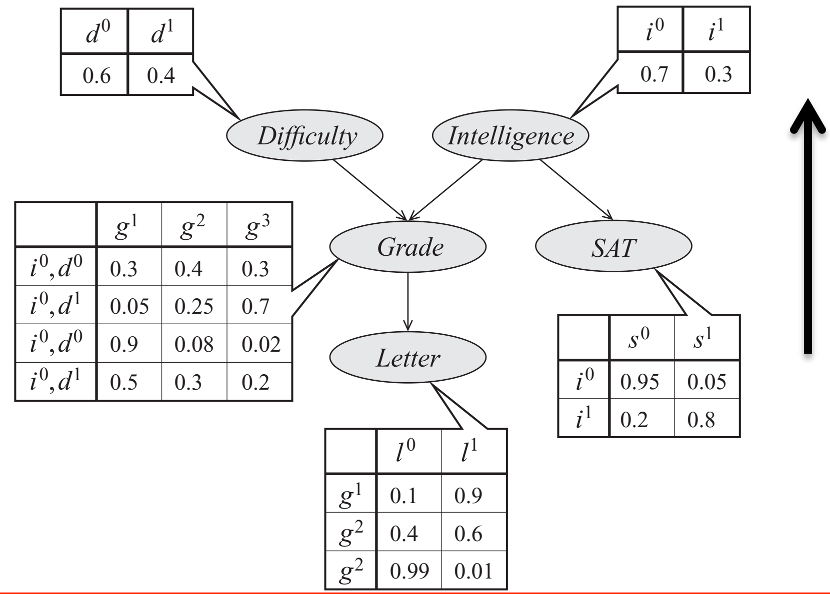
Query is Example of Causal Reasoning:

Predicting downstream effects of factors such as intelligence

Evidential Reasoning

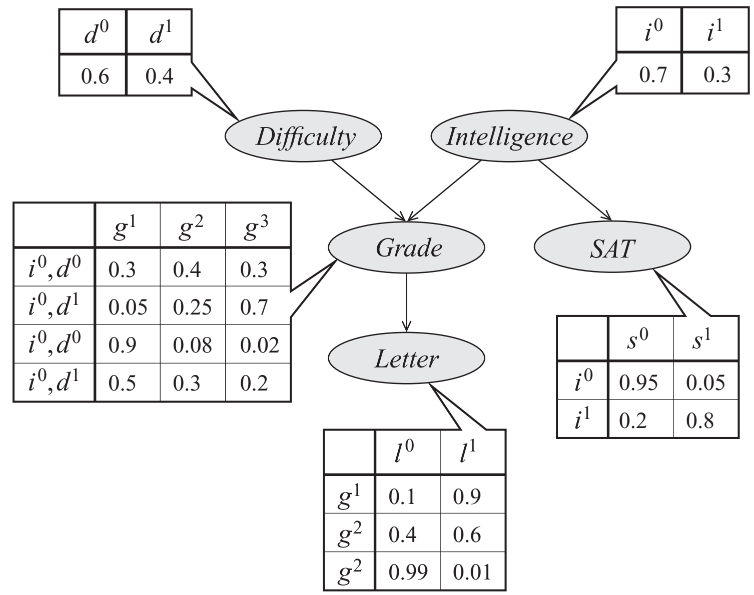
- Recruiter wants to hire intelligent student
- A priori George is 30% likely to be intelligent
- $P(i^1)=0.3$
- Finds that George received grade C (g^3) in ECON101
- $P(i^1|g^3)=0.079$
- Similarly probability class is difficult goes up from 0.4 to
- $P(d^1|g^3)=0.629$
- If recruiter has lost grade but has letter
- $P(i^1|l^0)=0.14$



- Recruiter has both grade and letter
- $P(i^1|l^0,g^3)=0.079$
  - Same as if he had only grade
  - Letter is immaterial
- Reasoning from effects to causes is called evidential reasoning

Intercausal reasoning

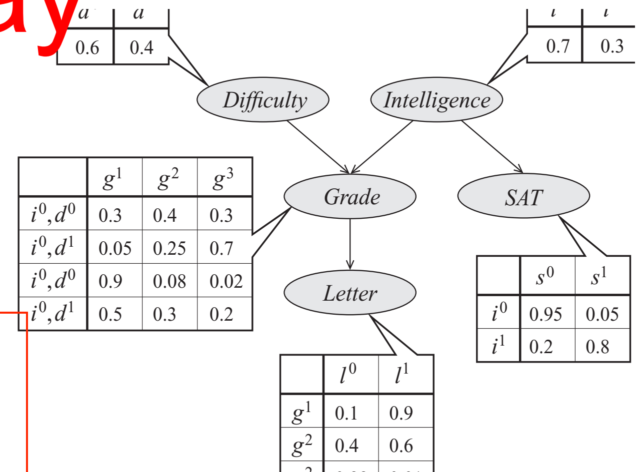
- Recruiter has grade (letter does not matter)
- $P(i^1|g^3)=P(i^1|l^0,g^3)=0.079$
- Recruiter receives high SAT score (leads to dramatic increase)
- $P(i^1|g^3,s^1)=0.578$
- Intuition:
  - High SAT score outweighs poor grade since low intelligence rarely gets good SAT scores
  - Smart students more likely to get Cs in hard classes
- Probability of class is difficult also goes up from
- $P(d^1|g^3)=0.629$ to
- $P(d^1|g^3,s^1)=0.76$



Information about SAT score gave us information about Intelligence which with Grade told us about difficulty of course

<u>One causal factor for Grade (Intelligence) gives us information about another (Difficulty)</u>

Explaining Away



An example:

- Given grade
- $P(i^1|l^0,g^3)=0.079$
- If we observe ECON101 is a hard class
- $P(i^1|g^3,d^1)=0.11$
- We have provided partial explanation for George's performance in ECON101

Another example:

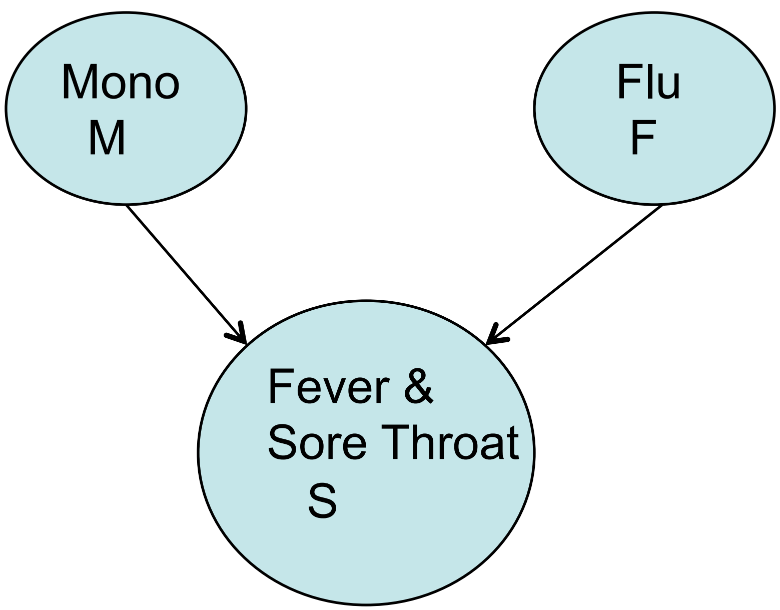
- If George gets a B in ECON101
- $P(i^1|g^2)=0.175$
- If we observe ECON101 is a hard class
- $P(i^1|g^2,d^1)=0.34$
- We have explained away the poor grade via the difficulty of the class

Explaining away is one type of intercausal reasoning

- Different causes of the same effect can interact
- All determined by probability calculation rather than heuristics

Intercausal Reasoning is Common in Human Reasoning

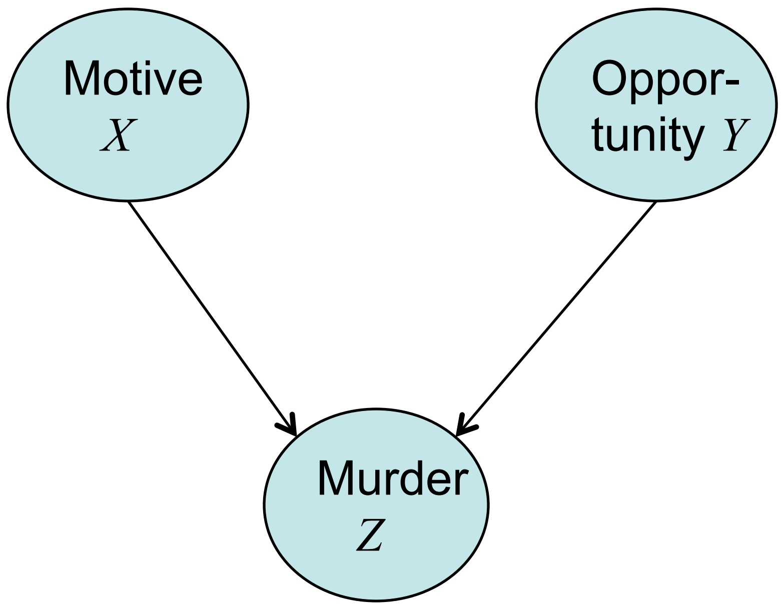
Another example of explaining away



- Binary Variables
- Fever & Sore Throat can be caused by mono and flu
- When flu is diagnosed probability of mono is reduced (although mono could still be present)
- It provides an alternative explanation of symptoms
- $P(m^1|s^1)>P(m^1|s^1,f^1)$

Another Type of Intercausal Reasoning

- Binary Variables
  - Murder (leaf node)
  - Motive and Opportunity are causal nodes
- Binary Variables X,Y,Z
- X and Y both increase the probability of Murder
  - $P(z^1|x^1)>P(z^1)$
  - $P(z^1|y^1)>P(z^1)$
- Each of X and Y increase probability of other
  - $P(x^1>z^1)<P(x^1|y^1,z^1)$
  - $P(y^1|z^1)<P(y^1|x^1,z^1)$



Can go in any direction

Different from Explaining Away

Dependencies and Independencies

- Crucial for understanding network behavior
- Independence properties are important for answering queries
  - Exploited to reduce computation of inference
  - A distribution P that factorizes over G satisfies $I(G)$
  - Are there other independencies that can be *read off* directly from G?
    - That hold for every P that factorizes over G

Conditional Independence

- Reminder: independence
  - X and Y are independent if

$$\forall x, y \;\; P(x,y) = P(x)P(y) \dashrightarrow X \perp\!\!\!\perp Y$$

  - X and Y are conditionally independent given Z

$$\forall x, y, z \;\; P(x,y|z) = P(x|z)P(y|z) \dashrightarrow X \perp\!\!\!\perp Y | Z$$

  - (Conditional) independence is a property of a distribution

Example: Independence

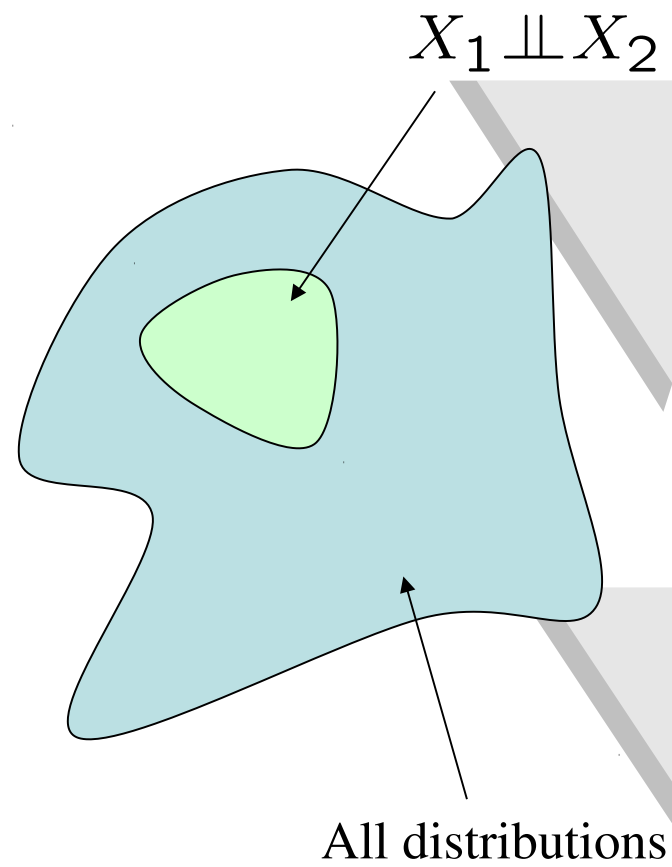
- For this graph, you can fiddle with θ (the CPTs) all you want, but you won't be able to represent any distribution in which the flips are dependent!

$P(X_1)$

h	0.5
t	0.5

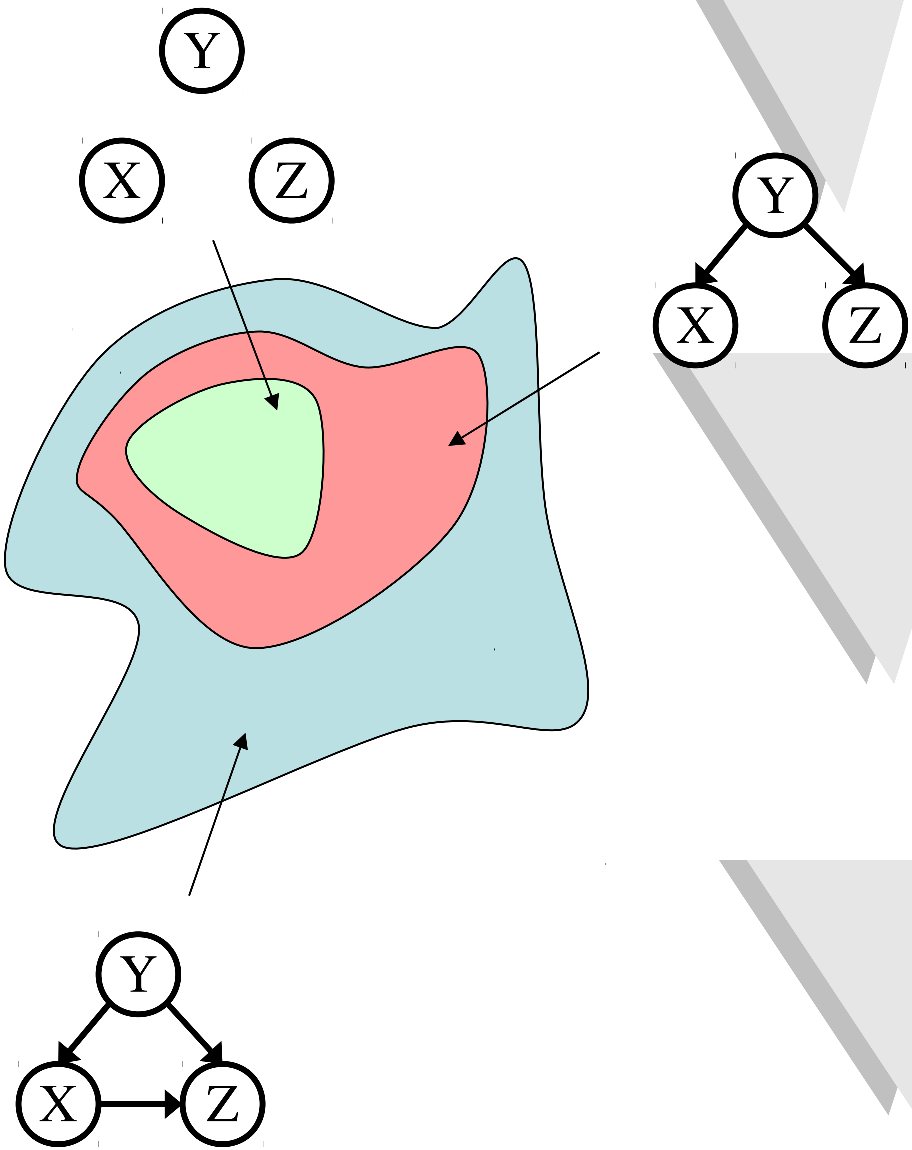
$P(X_2)$

h	0.5
t	0.5



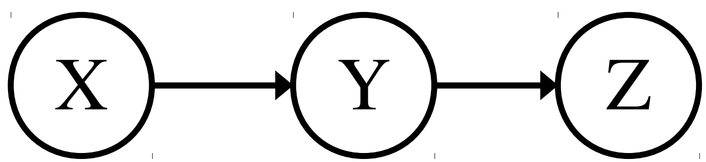
Topology Limits Distributions

- Given some graph topology G, only certain joint distributions can be encoded
- The graph structure guarantees certain (conditional) independences
- (There might be more independence)
- Adding arcs increases the set of distributions, but has several costs



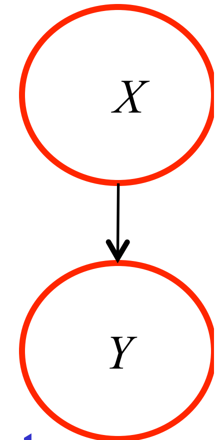
Independence in a BN

- Important question about a BN:
  - Are two nodes independent given certain evidence?
  - If yes, can calculate using algebra (really tedious)
  - If no, can prove with a counter example
  - Example:

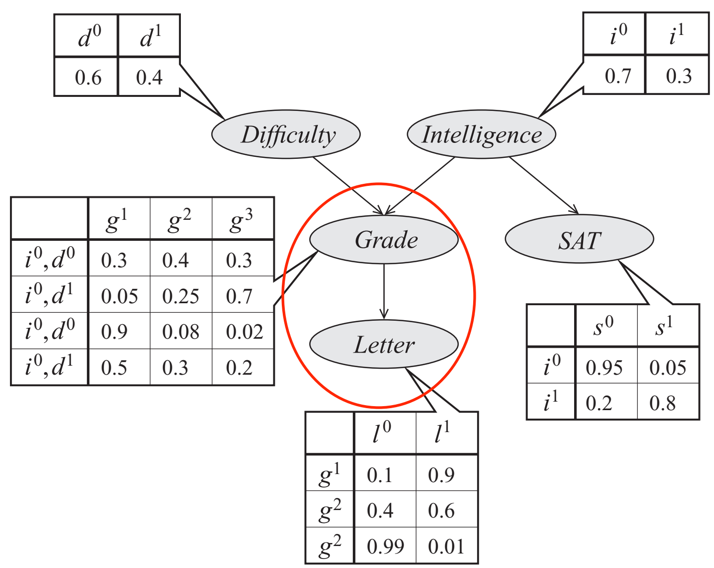


  - Question: are X and Z independent?
    - Answer: not *necessarily*, we've seen examples otherwise: low pressure causes rain which causes traffic.
    - X can influence Z, Z can influence X (via Y)
    - Addendum: they *could* be independent: how?

Direct Connection between X and Y



- X and Y are correlated regardless of any evidence about any other variables
  - E.g., Feature Y and character X are correlated
  - Grade G and Letter L are correlated
- If X and Y are directly connected we can get examples where they influence each other regardless of Z



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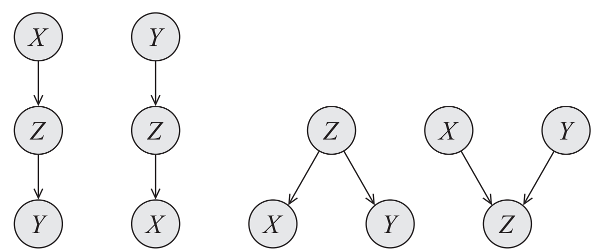
Indirect Connection betwn X and Y

- Four cases where X and Y are connected via Z
  1. Indirect causal effect
  2. Indirect evidential effect
  3. Common cause
  4. Common effect

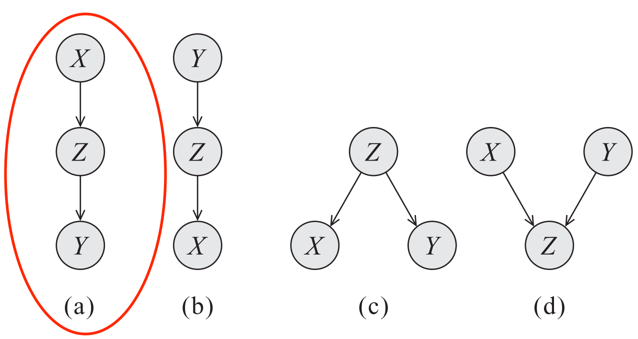


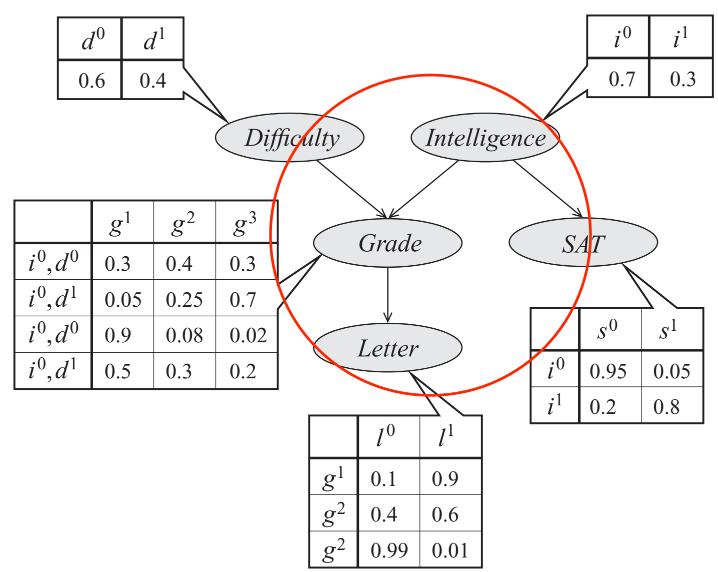
(a) (b) (c) (d)

- We will see that first three cases are similar while fourth case (V-structure) is different

1. Indirect Causal Effect: $X \rightarrow Z \rightarrow Y$

- Cause X cannot influence effect Y if Z observed
  – Observed Z blocks influence
  - If Grade is observed then I does not influence L
  – Intell influences Letter if Grade is unobserved



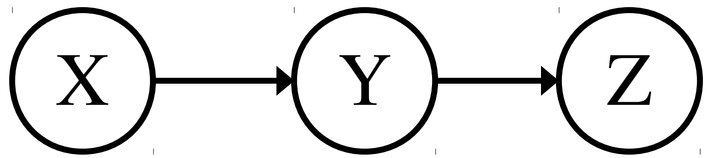


Z = Grade

I ind L|G

Causal Chains

- This configuration is a "causal chain"



X: Low pressure

Y: Rain

Z: Traffic

$$P(x, y, z) = P(x)P(y|x)P(z|y)$$

- Is X independent of Z given Y?

$$P(z|x, y) = \frac{P(x, y, z)}{P(x, y)} = \frac{P(x)P(y|x)P(z|y)}{P(x)P(y|x)}$$

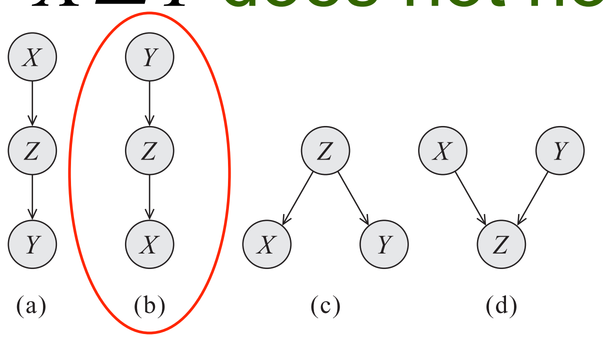
$$= P(z|y)$$

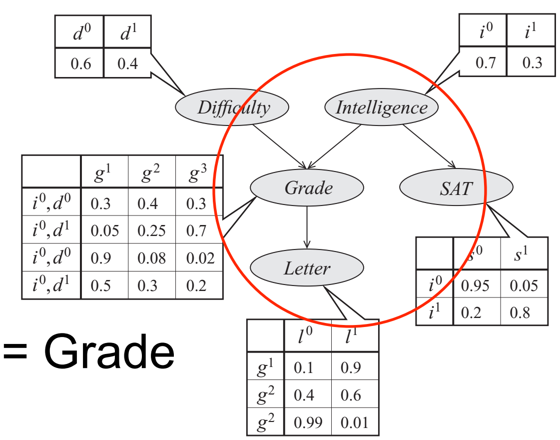
Yes!

- Evidence along the chain "blocks" the influence

2. Indirect Evidential Effect: $Y \rightarrow Z \rightarrow X$

- Evidence X can influence Y via Z only if Z is unobserved
  - Observed Z blocks influence
- If Grade unobserved, Letter influences assessment of Intelligence
- Dependency is a symmetric notion
  - $X \perp Y$ does not hold then $Y \perp X$ does not hold either

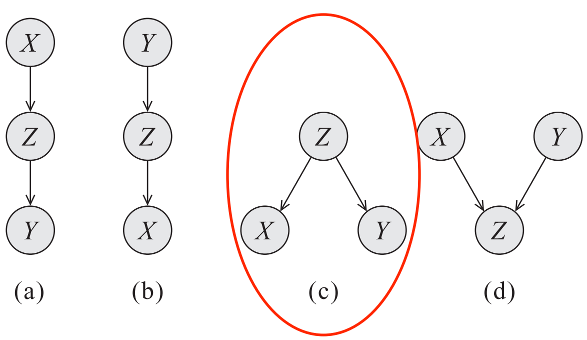


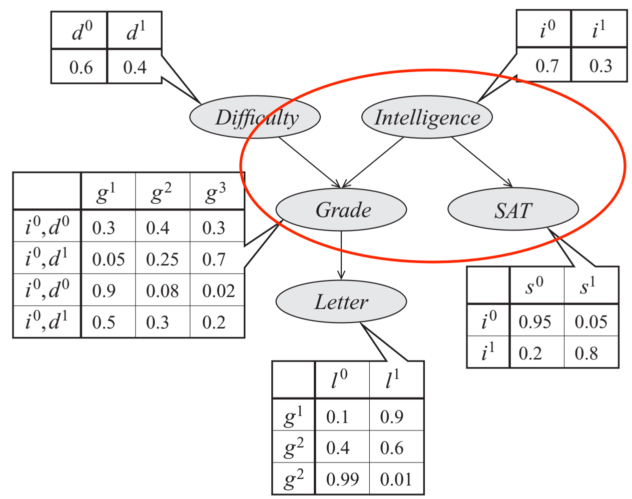


Z = Grade

3. Common Cause: $X \leftarrow Z \rightarrow Y$

- X can influence Y if and only if Z is not observed
  - Observed Z blocks
- Grade is correlated with SAT score
- But if Intelligence is observed then SAT provides no additional information





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S ind G|I

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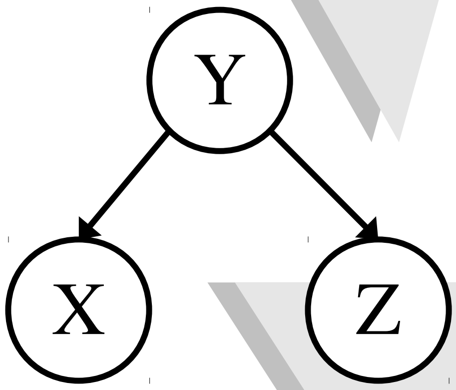
Common Cause

- Another basic configuration: two effects of the same cause
  - Are X and Z independent?
  - Are X and Z independent given Y?



Y: Project due

X: Email busy

Z: Lab full

$$P(z|x,y) = \frac{P(x,y,z)}{P(x,y)} = \frac{P(y)P(x|y)P(z|y)}{P(y)P(x|y)}$$

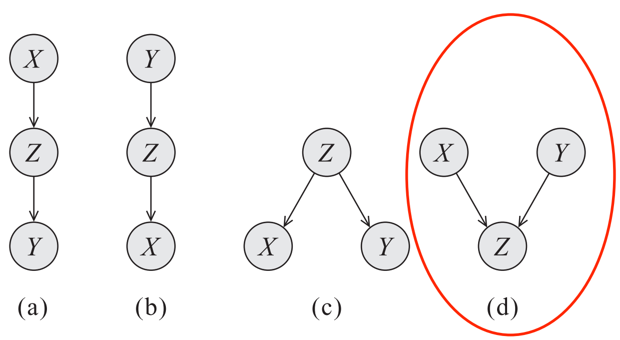
$$= P(z|y)$$

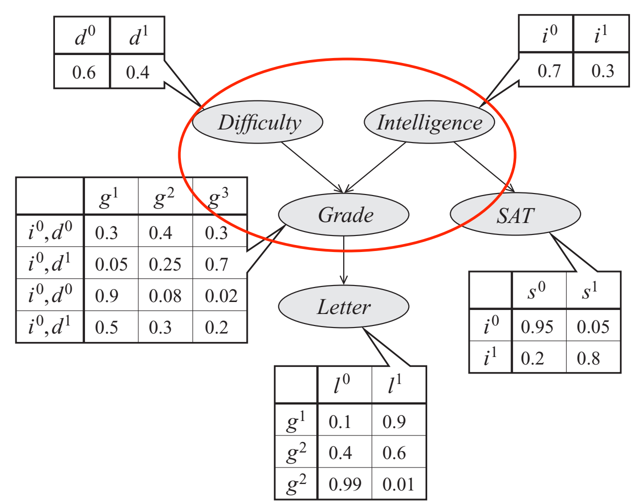
Yes!

  - Observing the cause blocks influence between effects.

4. Common Effect (V-structure) $X \rightarrow Z \leftarrow Y$

- Influence cannot flow on trail $X \rightarrow Z \leftarrow Y$ if Z is not observed
  - Observed Z enables
  - Opposite to previous 3 cases (Observed Z blocks)
- When G not observed I and D are independent
- When G is observed, I and D are correlated

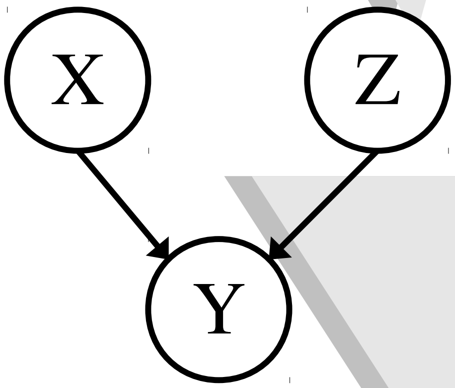




I ind D|~G

Common Effect

- Last configuration: two causes of one effect (v-structures)
  - Are X and Z independent?
    - Yes: remember the ballgame and the rain causing traffic, no correlation?
    - Still need to prove they must be (try it!)
  - Are X and Z independent given Y?
    - No: remember that seeing traffic put the rain and the ballgame in competition?
  - This is backwards from the other cases
    - Observing the effect enables influence between effects.

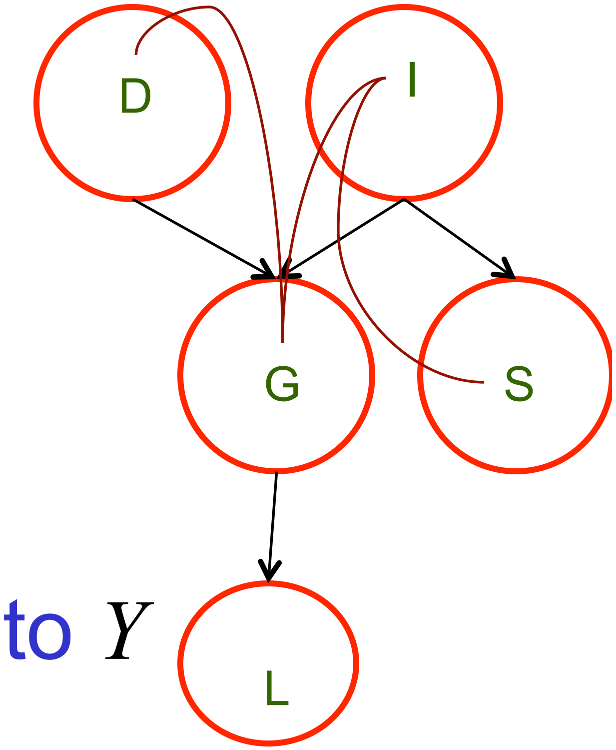


X: Raining

Z: Ballgame

Y: Traffic

Active Trail



- Grade is not observed
- Observe weak letter
  - Which indicates low grade
  - Suffices to correlate D and I
- When influence can flow from X to Y via Z then trail X—Z—Y is active
- Summary

Causal trail: $X \rightarrow Z \rightarrow Y$: active iff Z not observed

Evidential Trail: $X \leftarrow Z \leftarrow Y$: active iff Z is not observed

Common Cause: $X \leftarrow Z \rightarrow Y$: active iff Z is not observed

Common Effect: $X \rightarrow Z \leftarrow Y$: active iff either Z or one of its descendants is observed

Consider Trail D→G←I→S

Z={ϕ}: *inactive* because v-structure D→G←I is inactive

Z={L}: *active* (D→G←I active) since L is descendant of G

Z={L,I}: *inactive* because observing I blocks G←I→S.

D-separation definition

- Let $\boldsymbol{X}, \boldsymbol{Y}$ and $\boldsymbol{Z}$ be three sets of nodes in G.
- $\boldsymbol{X}$ and $\boldsymbol{Y}$ are d-separated given $\boldsymbol{Z}$ denoted $d\text{-}sep_G(\boldsymbol{X}:\boldsymbol{Y}|\boldsymbol{Z})$ if there is no active trail between any node $X \in \boldsymbol{X}$ and $Y \in \boldsymbol{Y}$ given $\boldsymbol{Z}$
- That is, nodes in $\boldsymbol{X}$ cannot influence nodes in $\boldsymbol{Y}$
- Provides notion of separation between nodes in a directed graph ("directed" separation)

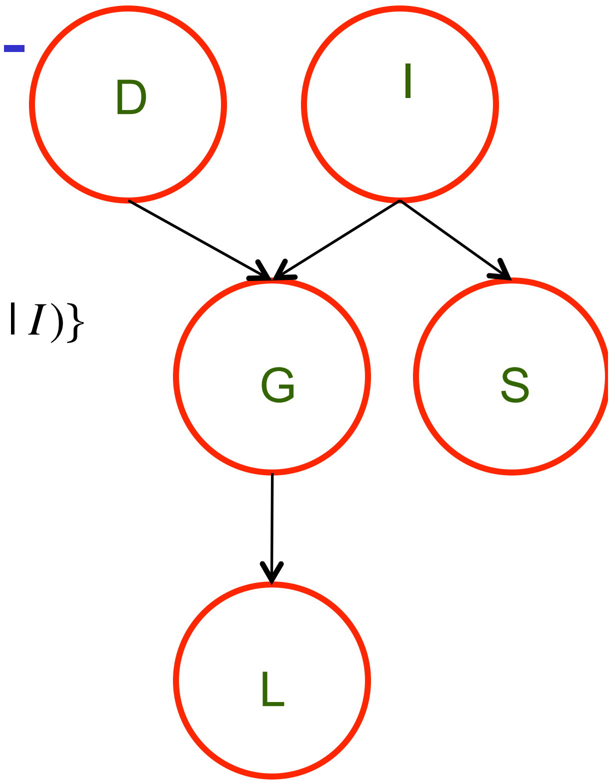
Independencies from d-separation

- Consider variables pairwise using d-separation

$$I(G)=\{(D\perp I,S,L\mid\phi),(\mathrm{I}\perp D,\mathrm{S},\mathrm{L}\mid\phi),$$
$$(G\perp L,S\mid D,I),(L\perp I,D,S\mid G),(S\perp D,G,L\mid I),(D\perp S\mid I)\}$$

  - Also called Markov independencies
- Definition:

$$I(G)=\{(X\perp Y|Z): d\text{-}sep_G(X:Y|Z)\}$$

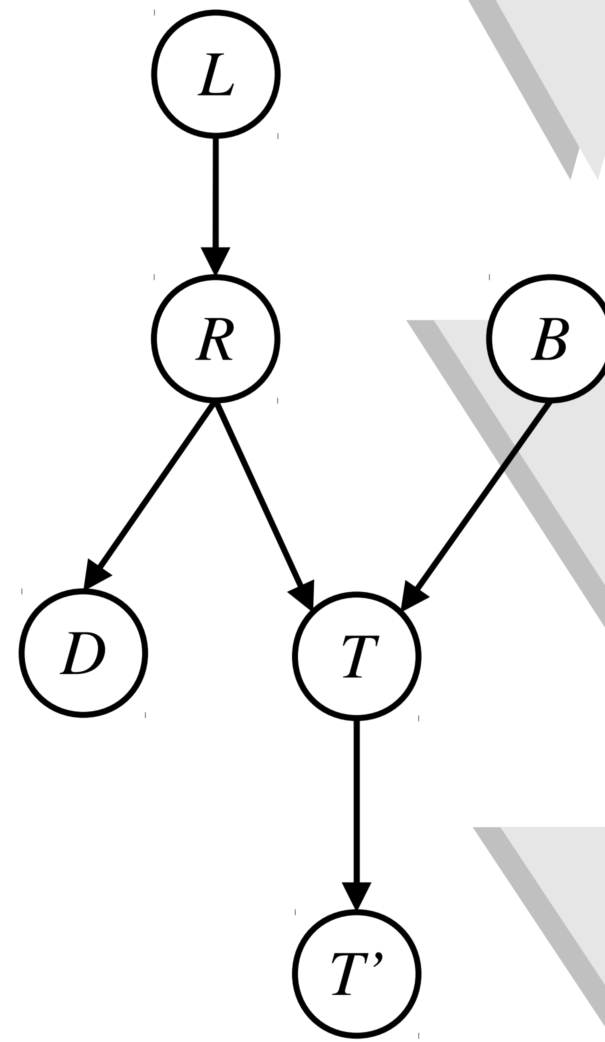


The General Case

- Any complex example can be analyzed using these three canonical cases
- General question: in a given BN, are two variables independent (given evidence)?
- Solution: analyze the graph

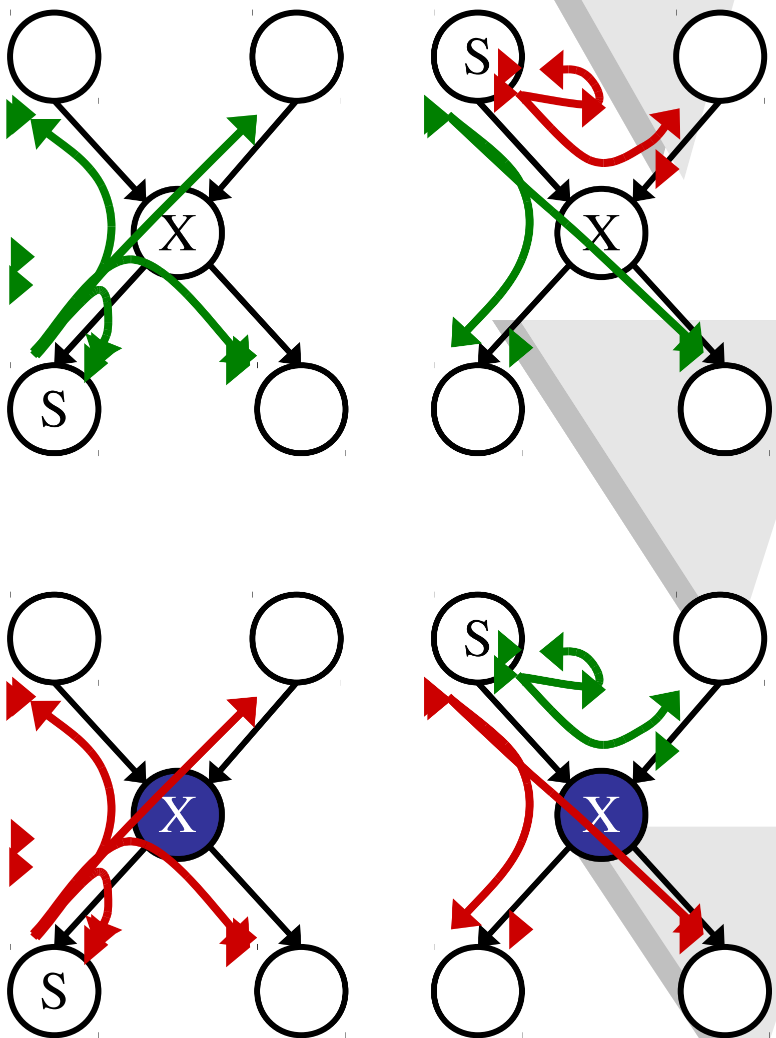
Reachability

- Recipe: shade evidence nodes
- Attempt 1: if two nodes are connected by an undirected path not blocked by a shaded node, they are conditionally independent
- Almost works, but not quite
  - Where does it break?
  - Answer: the v-structure at T doesn't count as a link in a path unless "active"



Reachability (the Bayes' Ball)

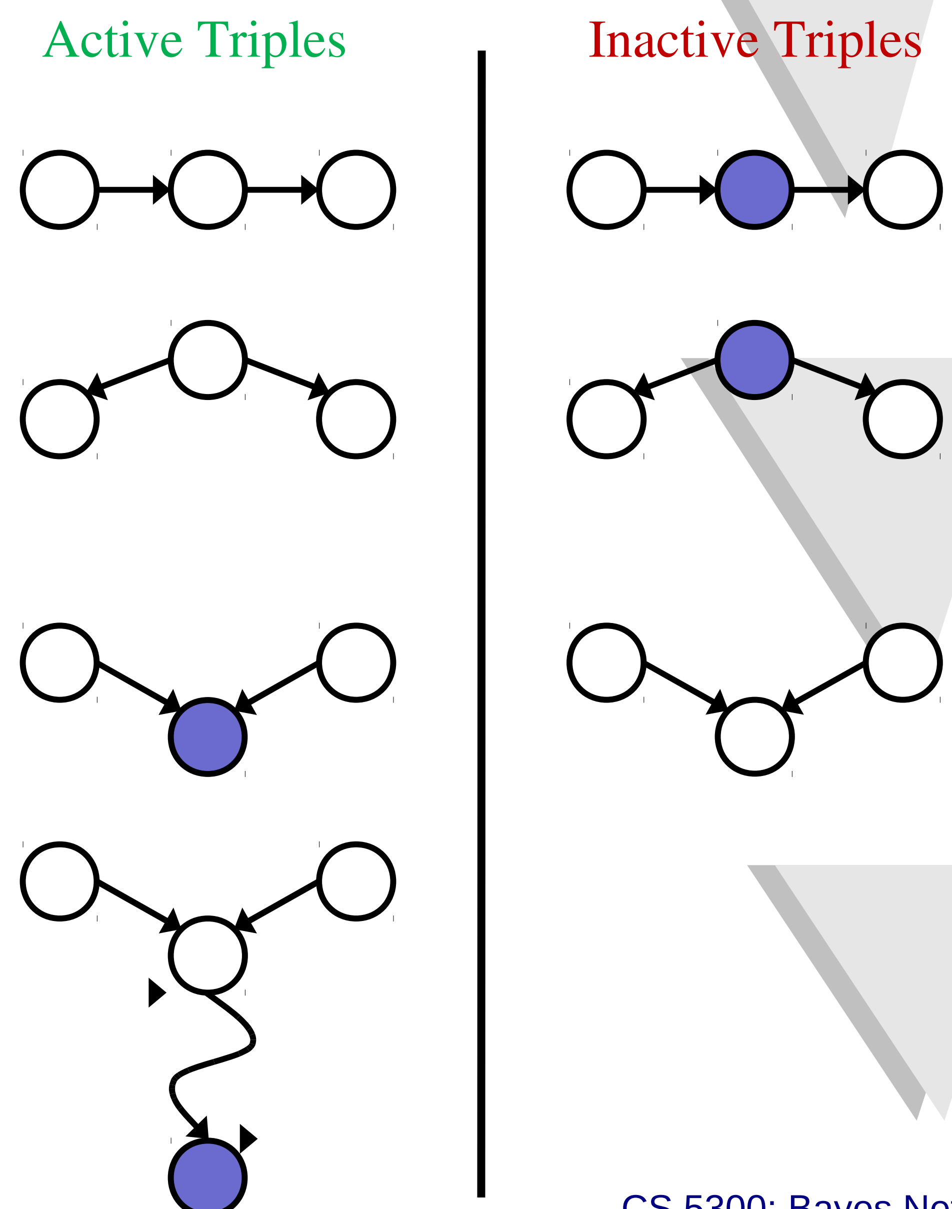
- Correct algorithm:
  - Shade in evidence
  - Start at source node
  - Try to reach target by search
  - States: pair of (node X, previous state S)
  - Successor function:
    - X unobserved:
      - To any child
      - To any parent if coming from a child
    - X observed:
      - From parent to parent
  - If you can't reach a node, it's conditionally independent of the start node given evidence



Reachability (D-Separation)

- Question: Are X and Y conditionally independent given evidence variables {Z}?
  - Look for “active paths” from X to Y
  - No active paths = independence!
- A path is active if each triple is either a:
  - Causal chain $A \rightarrow B \rightarrow C$ where B is unobserved (either direction)
  - Common cause $A \leftarrow B \rightarrow C$ where B is unobserved
  - Common effect (aka v-structure)

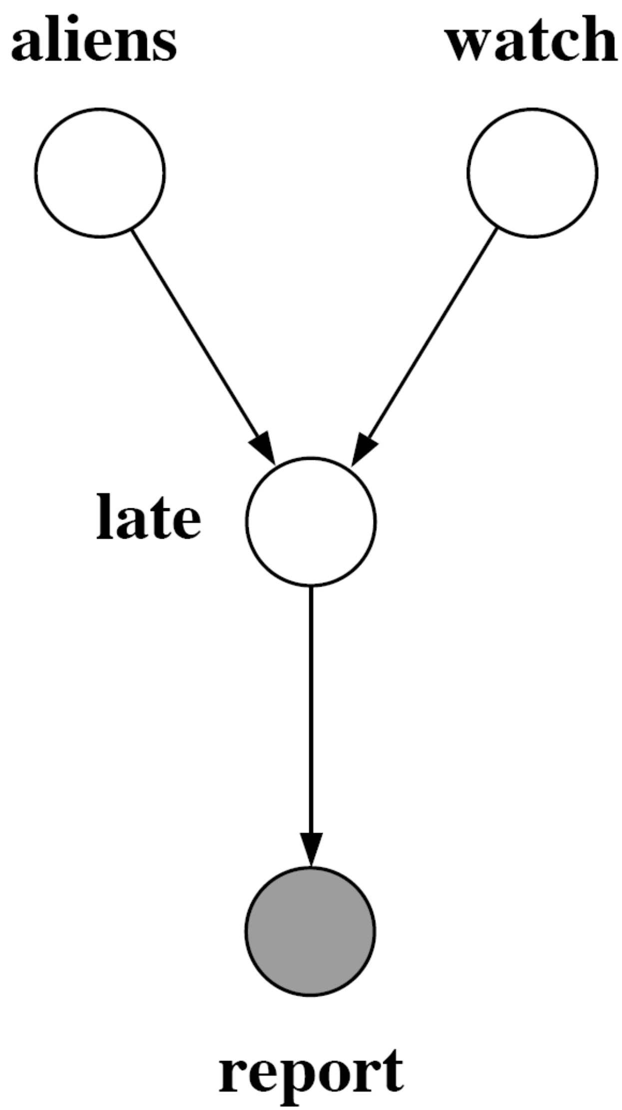
  $A \rightarrow B \leftarrow C$ where B or one of its descendents is observed



Example

$A \perp\!\!\!\perp W$ *Yes*

$A \perp\!\!\!\perp W \mid R$



Example

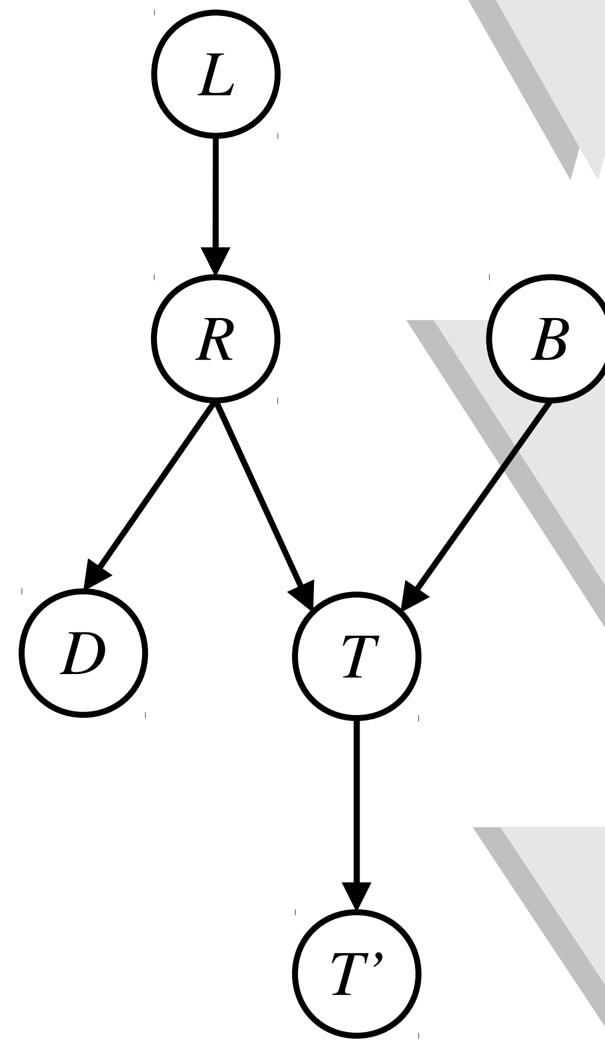
$L \perp\!\!\!\perp T' | T$ *Yes*

$L \perp\!\!\!\perp B$ *Yes*

$L \perp\!\!\!\perp B | T$

$L \perp\!\!\!\perp B | T'$

$L \perp\!\!\!\perp B | T, R$ *Yes*



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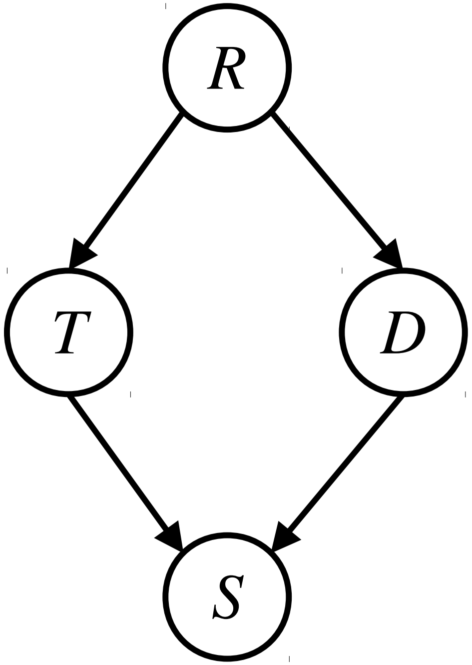
Example

- Variables:
  - R: Raining
  - T: Traffic
  - D: Roof drips
  - S: I'm sad
- Questions:



$$T \perp\!\!\!\perp D$$

$$T \perp\!\!\!\perp D | R$$ *Yes*

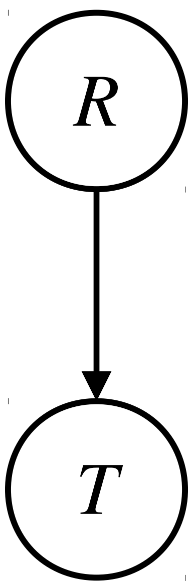
$$T \perp\!\!\!\perp D | R, S$$

Causality?

- When Bayes' nets reflect the true causal patterns:
  - Often simpler (nodes have fewer parents)
  - Often easier to think about
  - Often easier to elicit from experts
- BNs need not actually be causal
  - Sometimes no causal net exists over the domain
  - E.g. consider the variables *Traffic* and *Drips*
  - End up with arrows that reflect correlation, not causation
- What do the arrows really mean?
  - Topology may happen to encode causal structure
  - Topology only guaranteed to encode conditional independence

Example: Traffic

- Basic traffic net
- Let's multiply out the joint



$P(R)$

r	1/4
¬r	3/4

$P(T|R)$

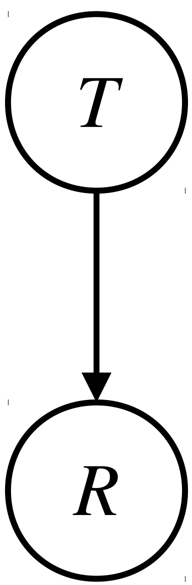
r	t	3/4
	¬t	1/4
¬r	t	1/2
	¬t	1/2

$P(T, R)$

r	t	3/16
r	¬t	1/16
¬r	t	6/16
¬r	¬t	6/16

Example: Reverse Traffic

- Reverse causality?



$P(T)$

t	9/16
¬t	7/16

$P(R|T)$

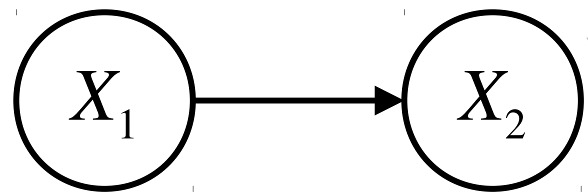
t	r	1/3
	¬r	2/3
¬t	r	1/7
	¬r	6/7

$P(T, R)$

r	t	3/16
r	¬t	1/16
¬r	t	6/16
¬r	¬t	6/16

Example: Coins

- Extra arcs don't prevent representing independence, just allow non-independence



$P(X_1)$

h	0.5
t	0.5

$P(X_2)$

h	0.5
t	0.5

$P(X_1)$

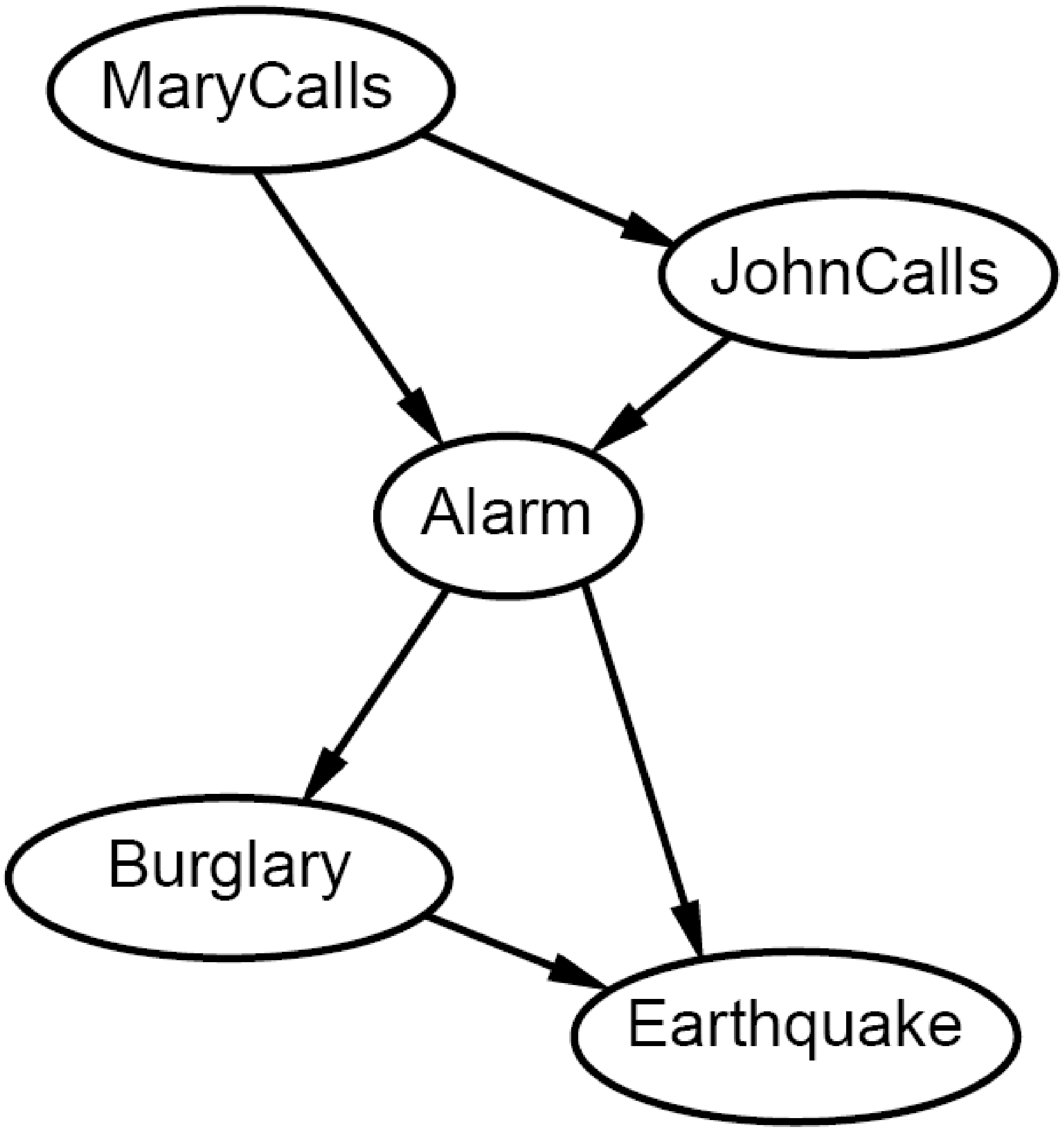
h	0.5
t	0.5

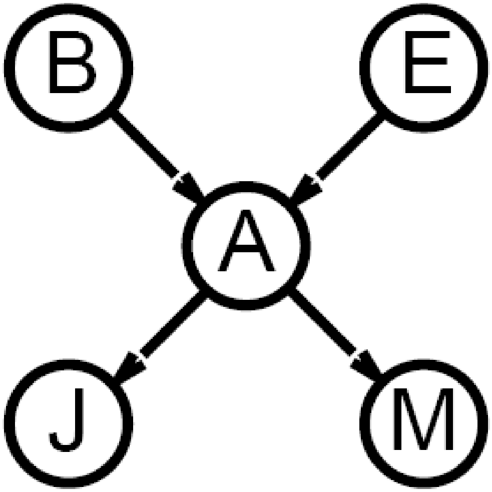
$P(X_2|X_1)$

h | h	0.5
t | h	0.5

h | t	0.5
t | t	0.5

Alternate BNs

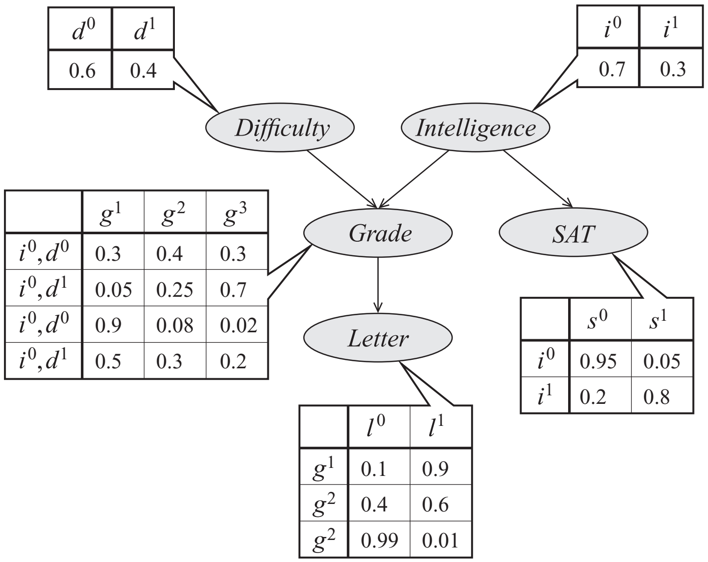




Summary

- Bayes nets compactly encode joint distributions
- Guaranteed independencies of distributions can be deduced from BN graph structure
- The Bayes' ball algorithm (aka d-separation)
- A Bayes' net may have other independencies that are not detectable until you inspect its specific distribution

Independencies in a BN



- Graph with CPDs
  is equivalent to a set of independence assertions

$$P(D,I,G,S,L) = P(D)P(I)P(G \mid D,I)P(S \mid I)P(L \mid G)$$

- Local Conditional Independence Assertions (starting from leaf nodes):

$I(G) = \{(L \perp I,D,S \mid G),$	L is conditionally independent of all other nodes given parent G
$(S \perp D,G,L \mid I),$	S is conditionally independent of all other nodes given parent I
$(G \perp S \mid D,I),$	Even given parents, G is NOT independent of descendant L
$(\mathrm{I} \perp D \mid \phi),$	Nodes with no parents are marginally independent
$(D \perp I,S \mid \phi)\}$	D is independent of non-descendants I and S

- Parents of a variable shield it from probabilistic influence
  - Once value of parents known, no influence of ancestors
- Information about descendants can change beliefs about a node

Soundness and Completeness

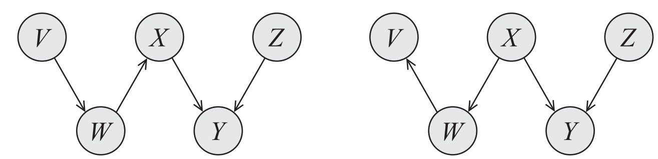
- Formalizing notion of d-separation
- Soundness Theorem
  - If a distribution P factorizes according to G then $I(G) \subseteq I(P)$
- A distribution P is faithful to graph G if any independence in P is reflected in G
  - G is then called a Perfect Map
- Completeness Theorem
  - Definition of I(G) is the maximal one
- Thus d-separation test precisely characterizes independencies that hold for P

Algorithm for d-separation

- Enumerating all trails is inefficient
  - Number of trails is exponential with graph size
- Linear time algorithm has two phases
- Algorithm $Reachable(G,X,\boldsymbol{Z})$ returns nodes for X
- Phase 1 (simple)
  - Traverse bottom-up from leaves marking all nodes in $\boldsymbol{Z}$ or descendants in $\boldsymbol{Z}$; to enable v-structures
- Phase 2 (subtle)
  - Traverse top-down from X to Y stopping when blocked by a node

I-Equivalence

- Conditional assertion statements can be the same with different structures
- Two graphs K_1 and K_2 are I-equivalent if $I(K_1)=I(K_2)$
- Skeleton of a BN graph G is an undirected graph with an edge for every edge in G
- If two BN graphs have the same set of skeletons and v-structures then they are I-equivalent



Same skeleton
Same v-structure $X \rightarrow Y \leftarrow Z$