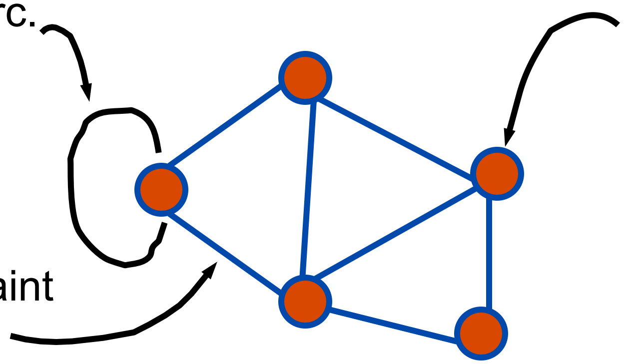
Constraint Satisfaction Problems

General class of Problems: Binary CSP

Unary constraint arc.

Variable V_i with values in domain D_i



Binary constraint arc

[Unary constraints just cut down domains]

This diagram is called a constraint graph

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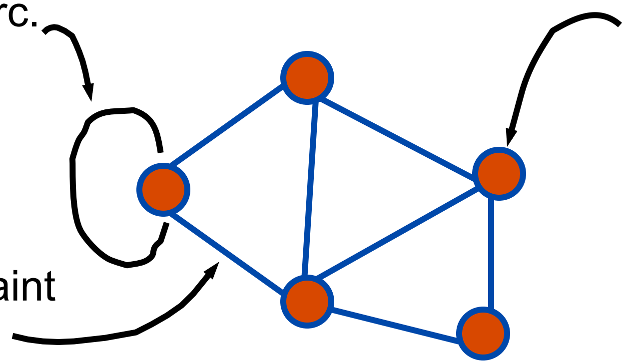


Binary constraint arc

Unary constraints just cut down domains

This diagram is called a constraint graph

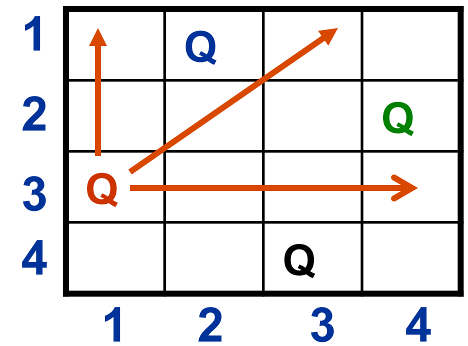
Basic problem:

Find a $d_j \in D_i$ for each V_i s.t. all constraints satisfied (finding consistent labeling for variables)

N-Queens as CSP

Classic “benchmark” problem

Place N queens on an NxN chessboard so that none can attack the other.

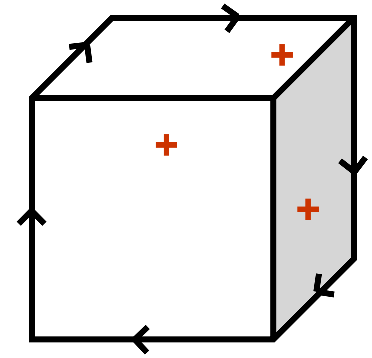


Variables are board positions in NxN chessboard

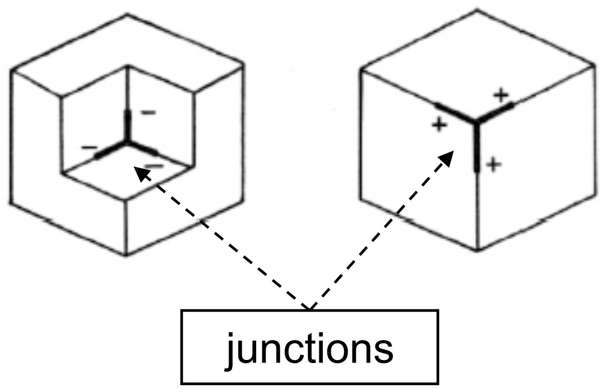
Domains Queen or blank

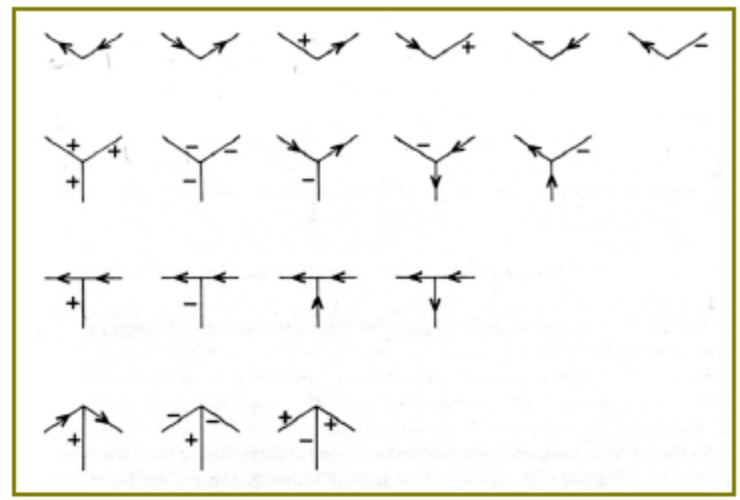
Constraints Two positions on a line (vertical, horizontal, diagonal) cannot both be Q

Line labelings as CSP



Label lines in drawing as convex (+), concave (-), or boundary (>).





All legal junction labels for four junction types

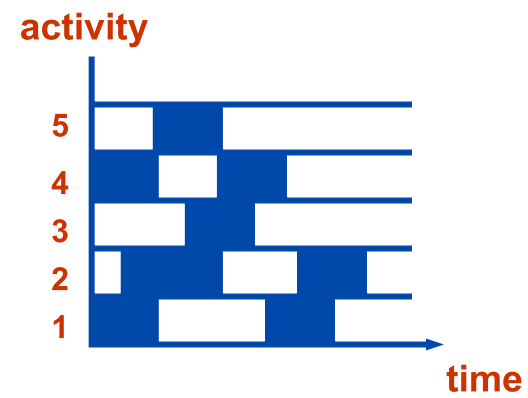
Variables are line junctions

Domains are set of legal labels for that junction type

Constraints shared lines between adjacent junctions must have same label.

Scheduling as CSP

Choose time for activities e.g. observations on Hubble telescope, or terms to take required classes.



Variables are activities

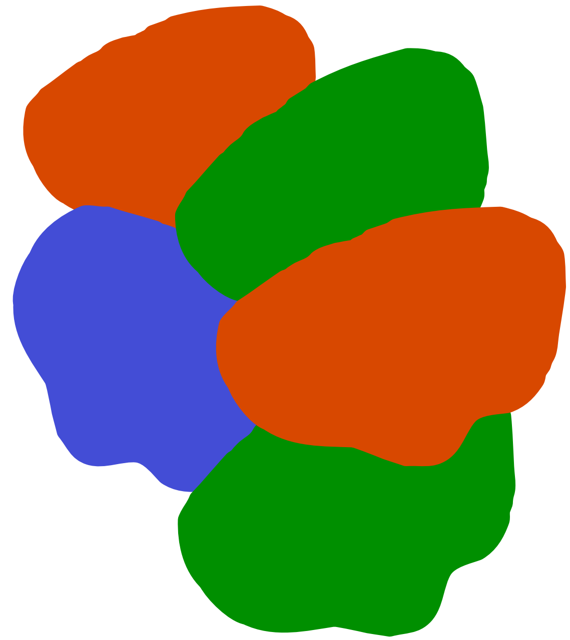
Domains sets of start times (or “chunks” of time)

Constraints

1. Activities that use same resource cannot overlap in time
2. Preconditions satisfied

Graph Coloring as CSP

Pick colors for map regions, avoiding coloring adjacent regions with the same color



Variables	regions
Domains	colors allowed
Constraints	adjacent regions must have different colors

3-SAT as CSP

The original NP-complete problem

(A or B or !C) and (!A or C or B) …

Find values for boolean variables A,B,C,… that satisfy the formula.

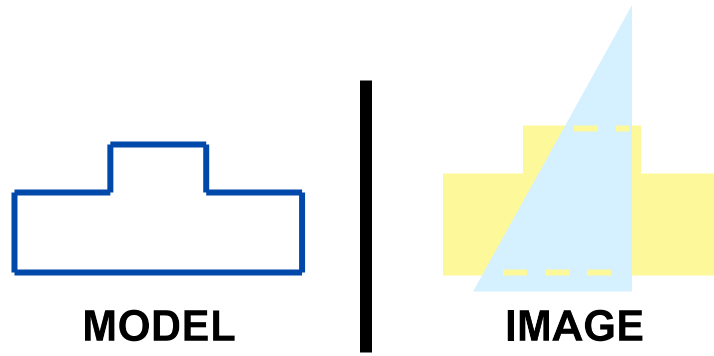
Variables clauses

Domains boolean variable assignments that make clause true

Constraints clauses with shared boolean variables must agree on value of variable

Model-based recognition as CSP

Find given model in edge image, with rotation and translation allowed.



Variables	edges in model
Domains	set of edges in image
Constraints	angle between model & image edges must match

Good News / Bad News

Good News - very general & interesting class problems

Bad News - includes NP-Hard (intractable) problems

So, good behavior is a function of domain not the formulation as CSP.

CSP Example

Given 40 courses (8.01, 8.02, ... 6.840) & 10 terms (Fall 1, Spring 1, ... , Spring 5). Find a legal schedule.

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- Avoid time conflicts

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Note, CSPs are not for expressing (soft) preferences e.g., minimize difficulty, balance subject areas, etc.

Choice of variables & values

VARIABLES

A. **Terms?**

DOMAINS

Legal combinations of for example 4 courses (but this is huge set of values).

Choice of variables & values

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B. **Term Slots?**	
subdivide terms into slots e.g. 4 of them (Fall 1,1) (Fall 1,2) (Fall1,3) (Fall 1,4)	Courses offered during that term

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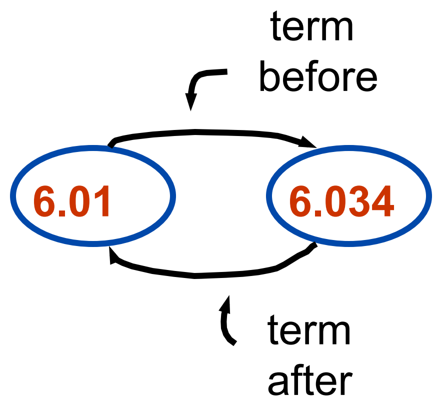
subdivide terms into slots e.g. 4 of them (Fall 1,1) (Fall 1,2) (Fall1,3) (Fall 1,4)	Courses offered during that term
C. **Courses?**	Terms or term slots (Term slots allow expressing constraint on limited number of of courses / term.)

Constraints

Use courses as variables and term slots as values.

Prerequisite ➡



For pairs of courses that must be ordered.

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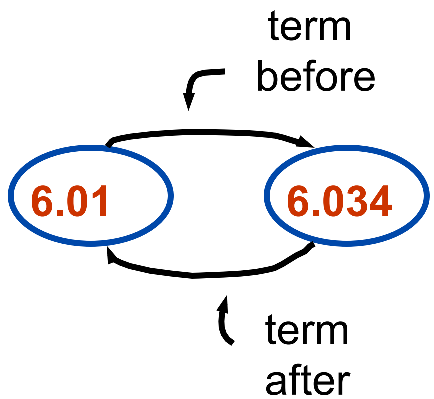

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Courses offered only in some terms ➡ **Filter domain**

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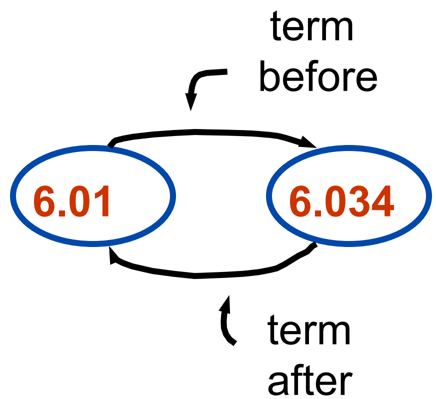

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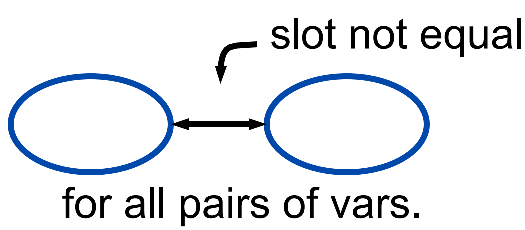
Limit # courses ➡



Use term-slots only once

Constraints

Use courses as variables and term slots as values.

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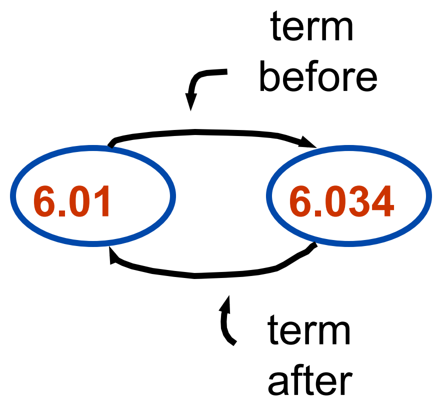

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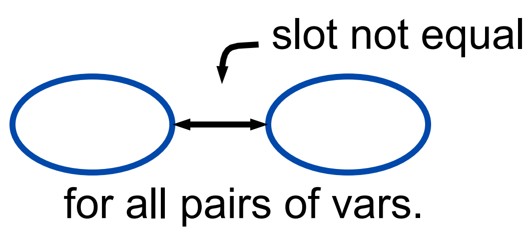

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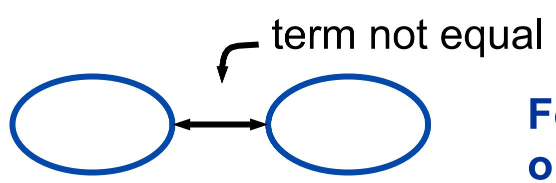
Avoid time conflicts ➡



For pairs offered at same or overlapping times

Solving CSPs

Solving CSPs involves some combination of:

1. Constraint propagation, to eliminate values that could not be part of any solution
2. Search, to explore valid assignments

Constraint Propagation (aka Arc Consistency)

Arc consistency eliminates values from domain of variable that can never be part of a consistent solution.

$$V_i \rightarrow V_j$$

Directed arc (V_i, V_j) is arc consistent if
$\forall x \in D_i \; \exists y \in D_j$ such that (x,y) is allowed by the constraint on the arc

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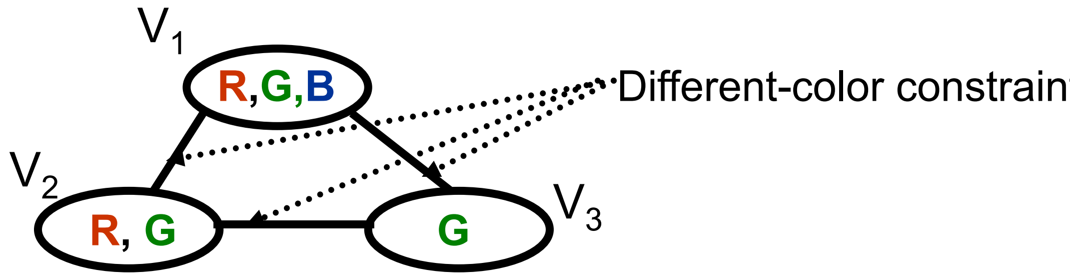
Assume domains are size at most <u>d</u> and there are <u>e</u> binary constraints.

A simple algorithm for arc consistency is $O(ed^3)$ – note that just verifying arc consistency takes $O(d^2)$ for each arc.

Constraint Propagation Example

Graph Coloring

Initial Domains are indicated



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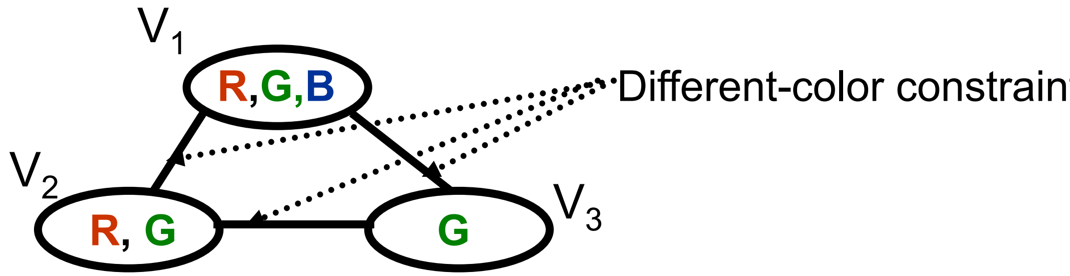

Graph Coloring

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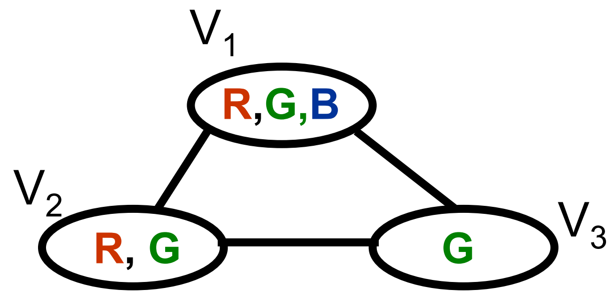
Arc examined	Value deleted



Each undirected constraint arc is really two directed constraint arcs, the effects shown above are from examining BOTH arcs.

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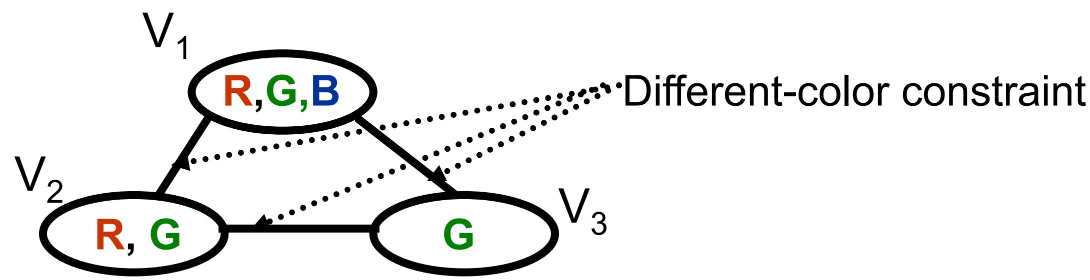
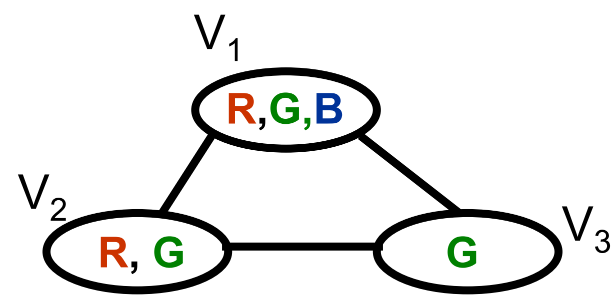


Arc examined	Value deleted
$V_1 – V_2$	none



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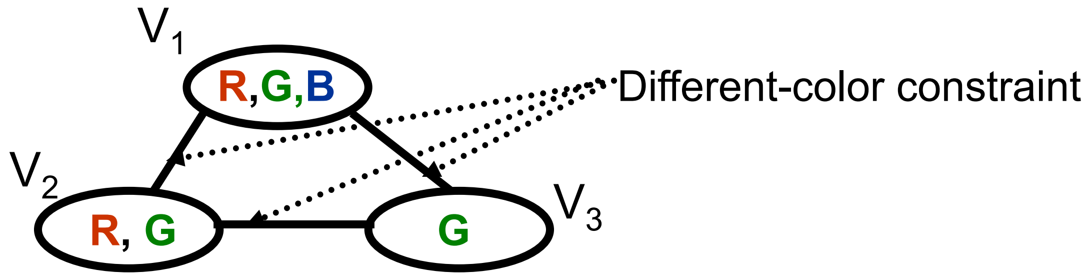
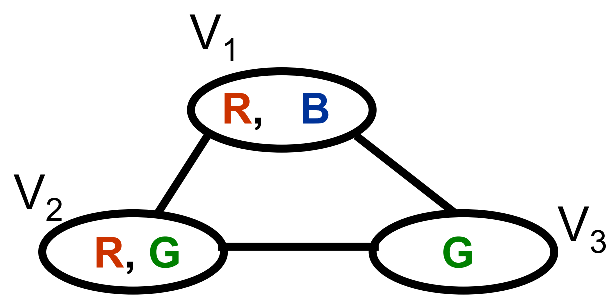


Arc examined	Value deleted
$V_1 - V_2$	none
$V_1 - V_3$	V_1(G)



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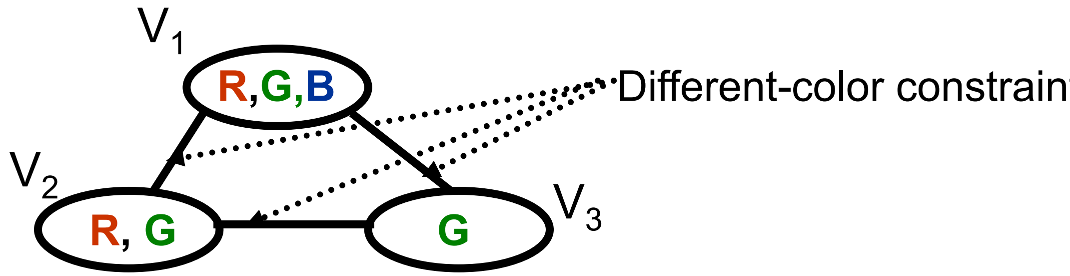
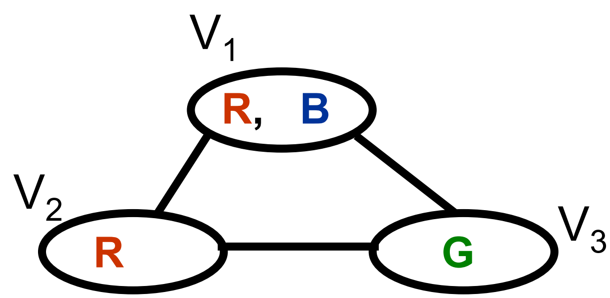


Arc examined	Value deleted
$V_1 - V_2$	none
$V_1 - V_3$	V_1(G)
$V_2 - V_3$	V_2(G)



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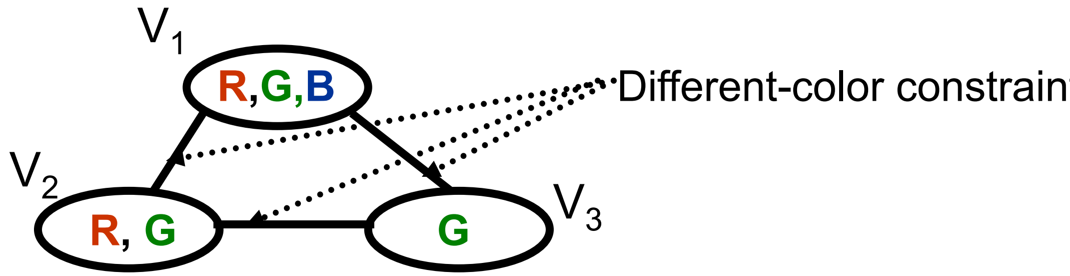
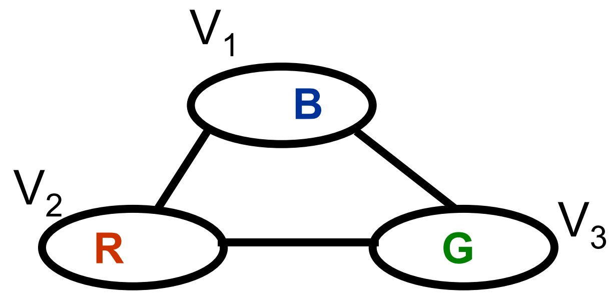


Arc examined	Value deleted
$V_1 - V_2$	none
$V_1 - V_3$	V_1(G)
$V_2 - V_3$	V_2(G)
$V_1 - V_2$	V_1(R)



Constraint Propagation Example

Graph Coloring

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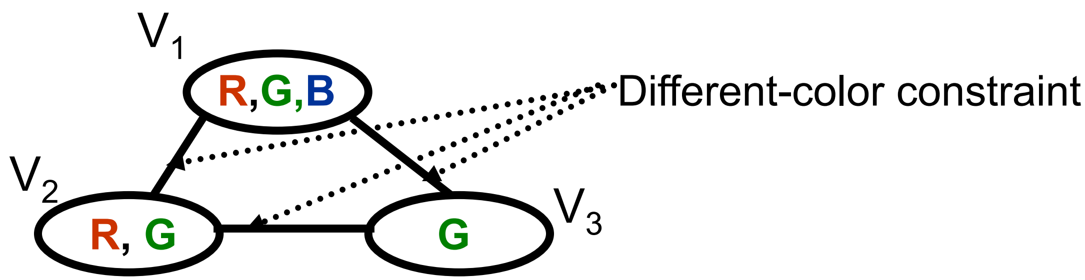
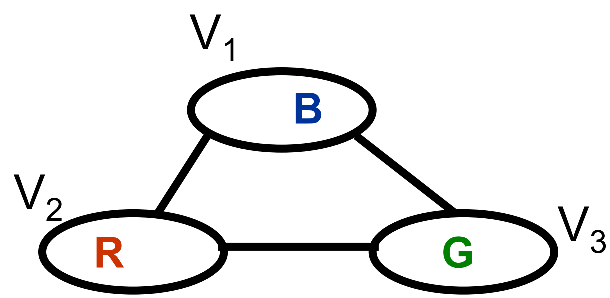


Arc examined	Value deleted
$V_1 - V_2$	none
$V_1 - V_3$	V_1(G)
$V_2 - V_3$	V_2(G)
$V_1 - V_2$	V_1(R)
$V_1 - V_3$	none



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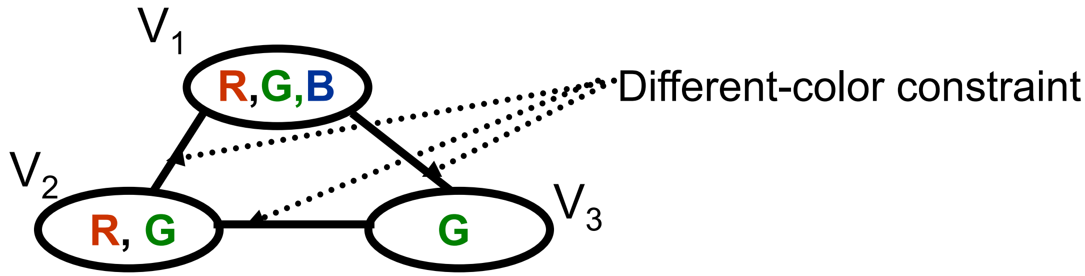

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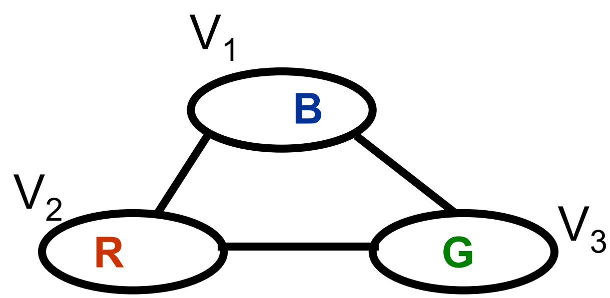
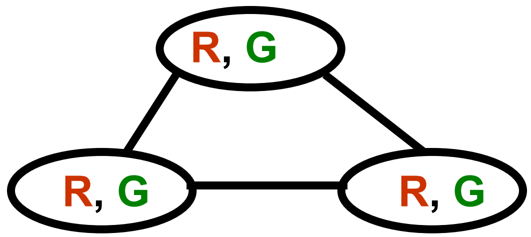


But, arc consistency is not enough in general

Graph Coloring



arc consistent but <u>no</u> solutions

Arc consistency algorithm

function AC-3(*csp*) **returns** the CSP, possibly with reduced domains
  inputs: *csp*, a binary CSP with variables $\{X_1, X_2, \ldots, X_n\}$
  local variables: *queue*, a queue of arcs, initially all the arcs in *csp*

  while *queue* is not empty **do**
    $(X_i, X_j) \leftarrow$ Remove-First(*queue*)
    if Remove-Inconsistent-Values(X_i, X_j) **then**
      for each X_k **in** Neighbors[X_i] **do**
        add (X_k, X_i) to *queue*

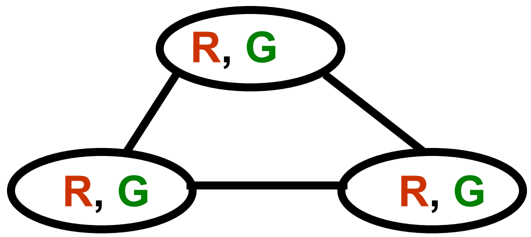
---

function Remove-Inconsistent-Values(X_i, X_j) **returns** true iff succeeds
  removed ← *false*
  for each x **in** Domain[X_i] **do**
    if no value y in Domain[X_j] allows (x,y) to satisfy the constraint $X_i \leftrightarrow X_j$
      then delete x from Domain[X_i]; *removed* ← *true*
  return *removed*

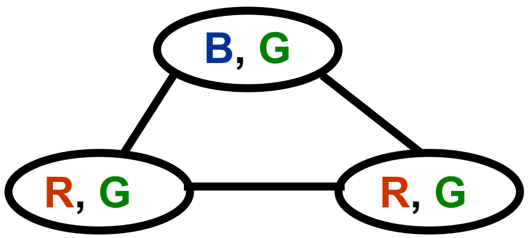
$O(n^2d^3)$, can be reduced to $O(n^2d^2)$ (but detecting **all** is NP-hard)

But, arc consistency is not enough in general

Graph Coloring



arc consistent but <u>no</u> solutions



arc consistent but <u>2</u> solutions B,R,G ; B,G,R .

But, arc consistency is not enough in general

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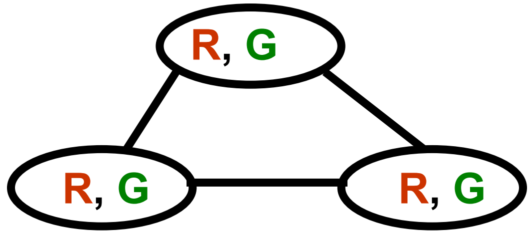
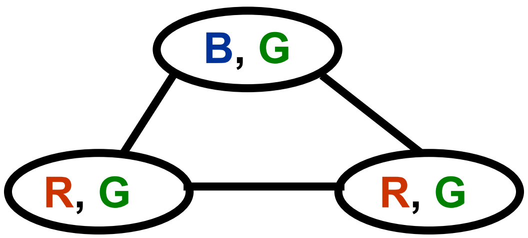


arc consistent but no solutions



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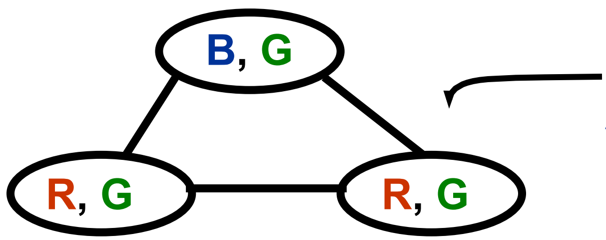
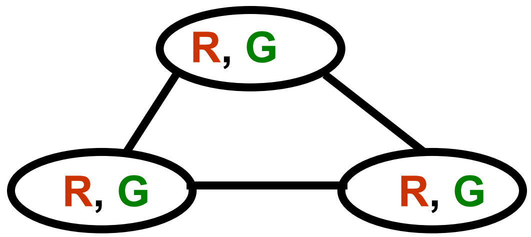


arc consistent but 1 solution

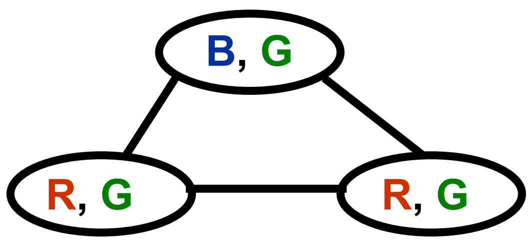
Assume B, R not allowed

But, arc consistency is not enough in general

Graph Coloring



arc consistent but <u>no</u> solutions



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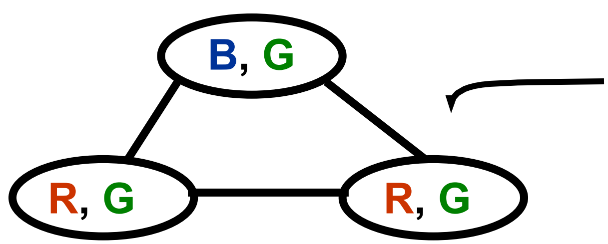


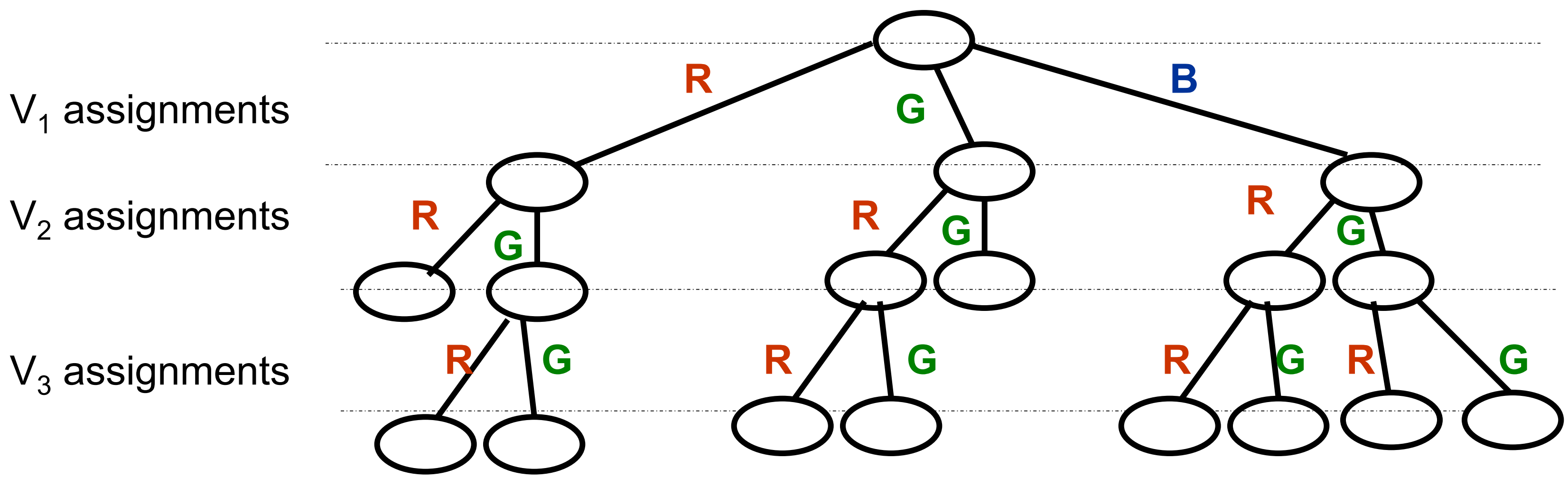
arc consistent but <u>1</u> solution

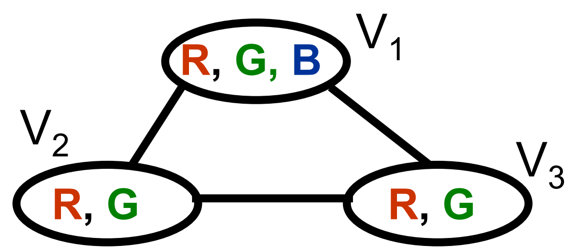
B, R not allowed

Need to do search to find solutions (if any)

Searching for solutions – backtracking (BT)

When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).





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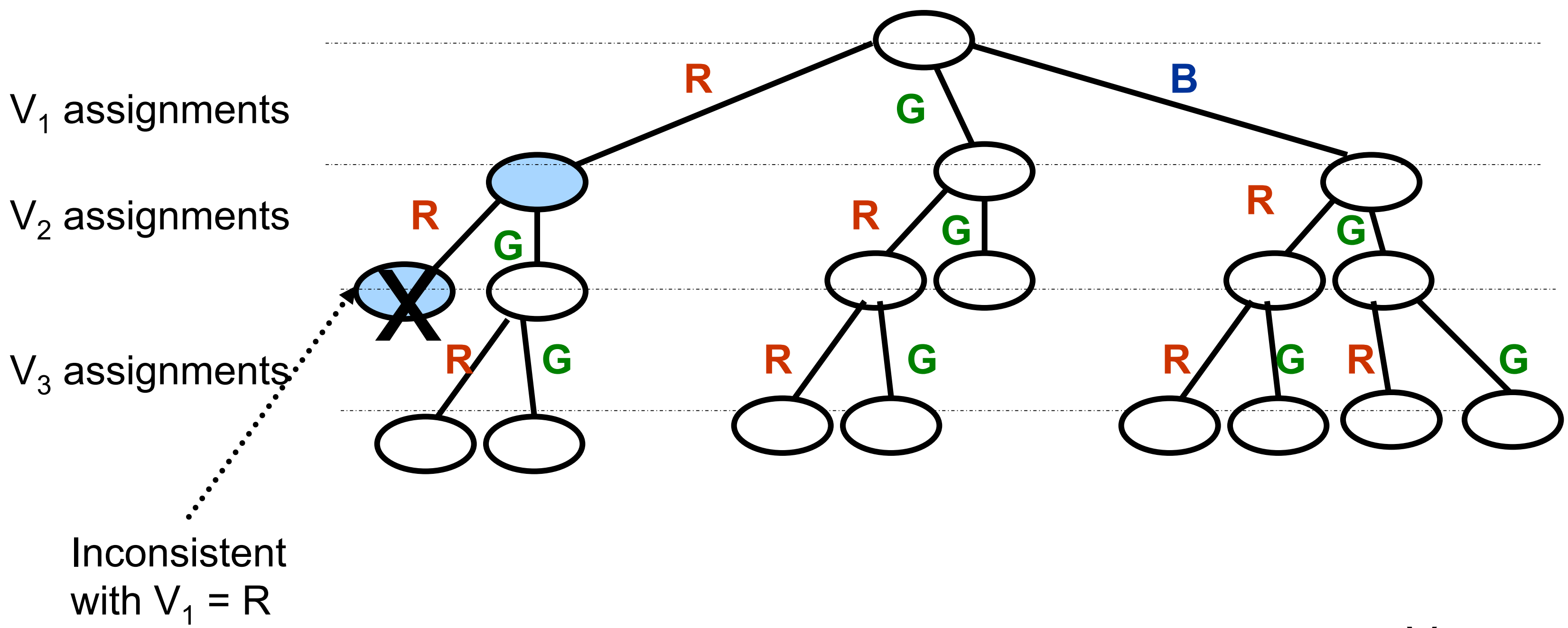
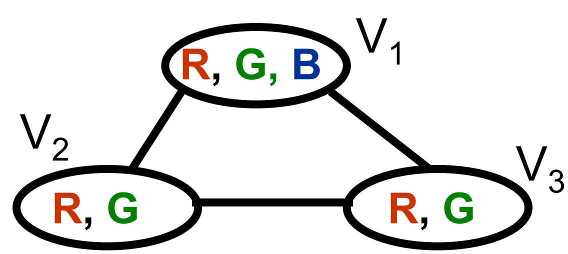


Backup at inconsistent assignment



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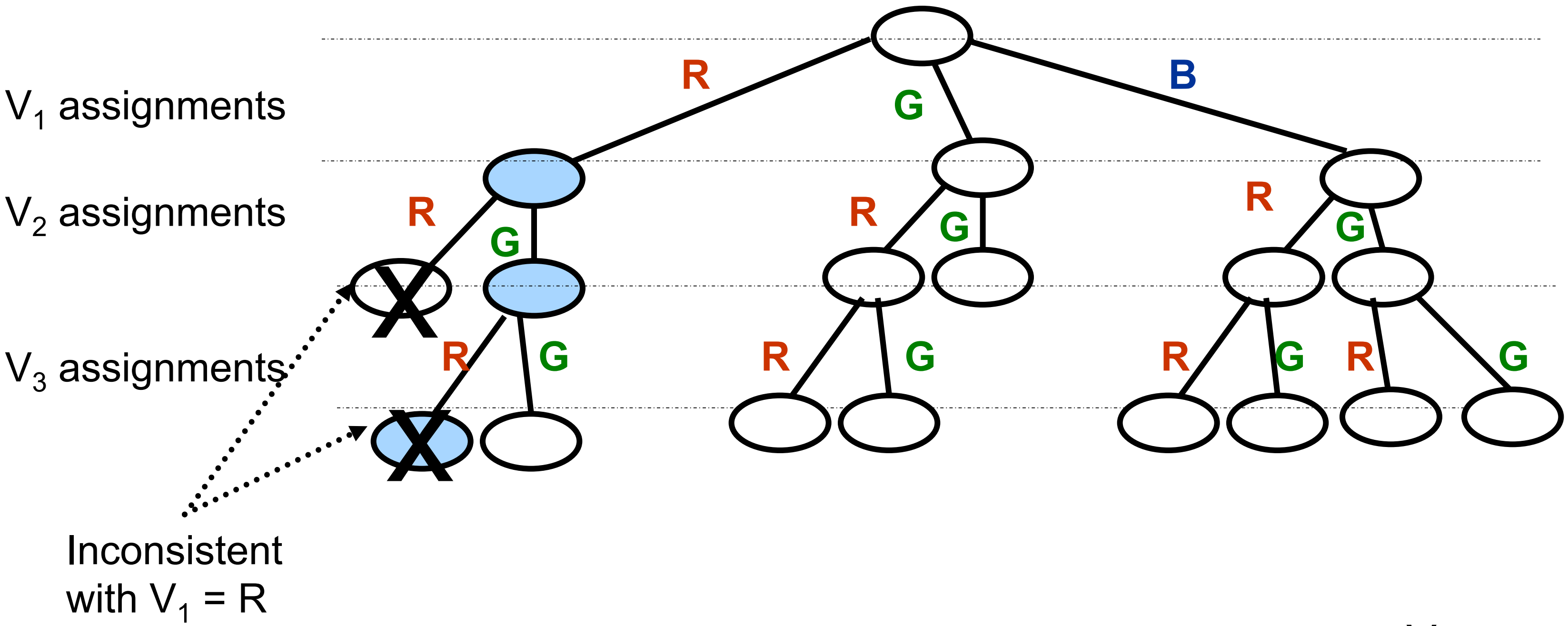

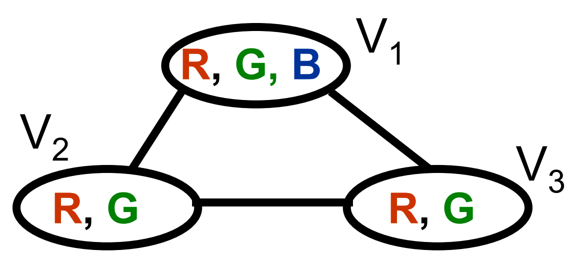

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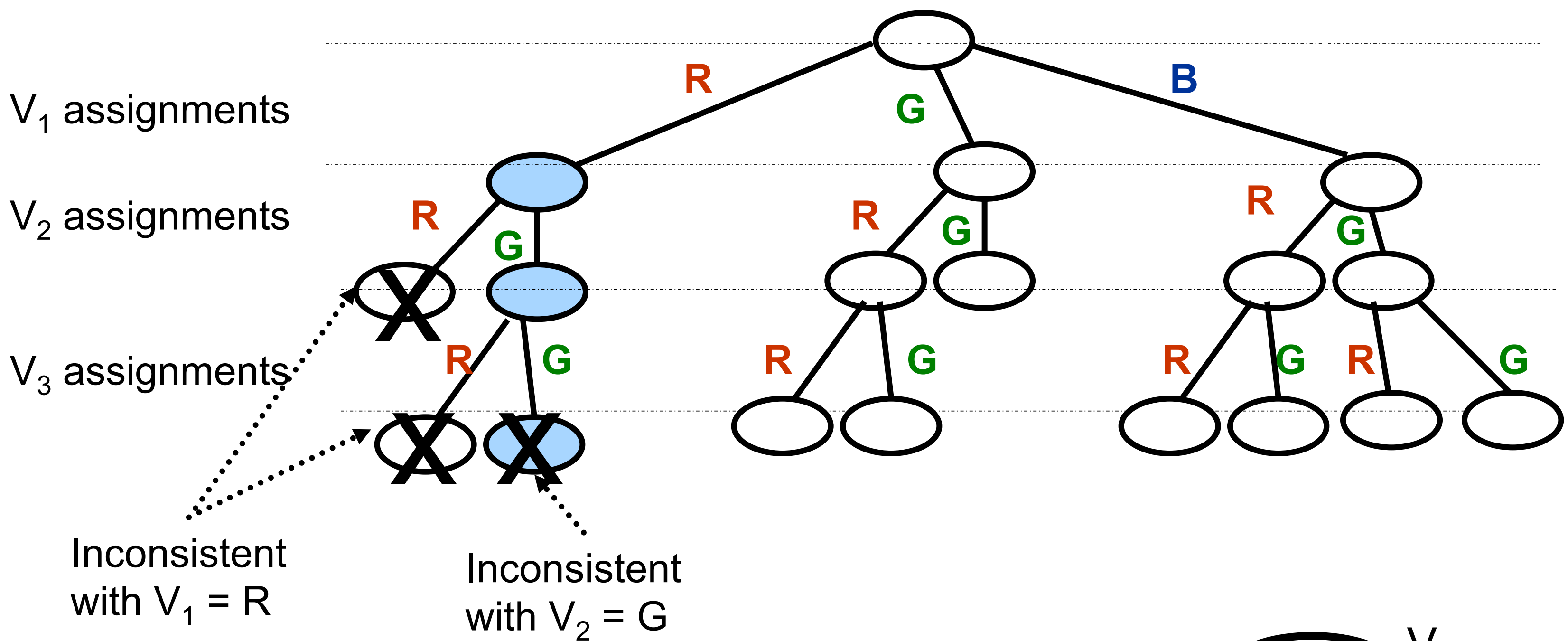

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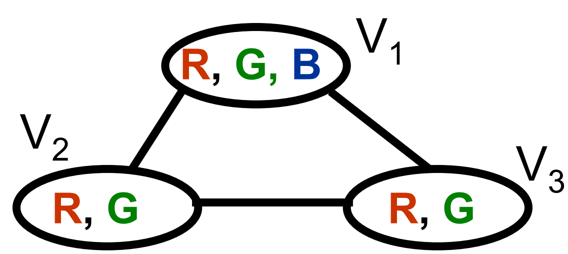
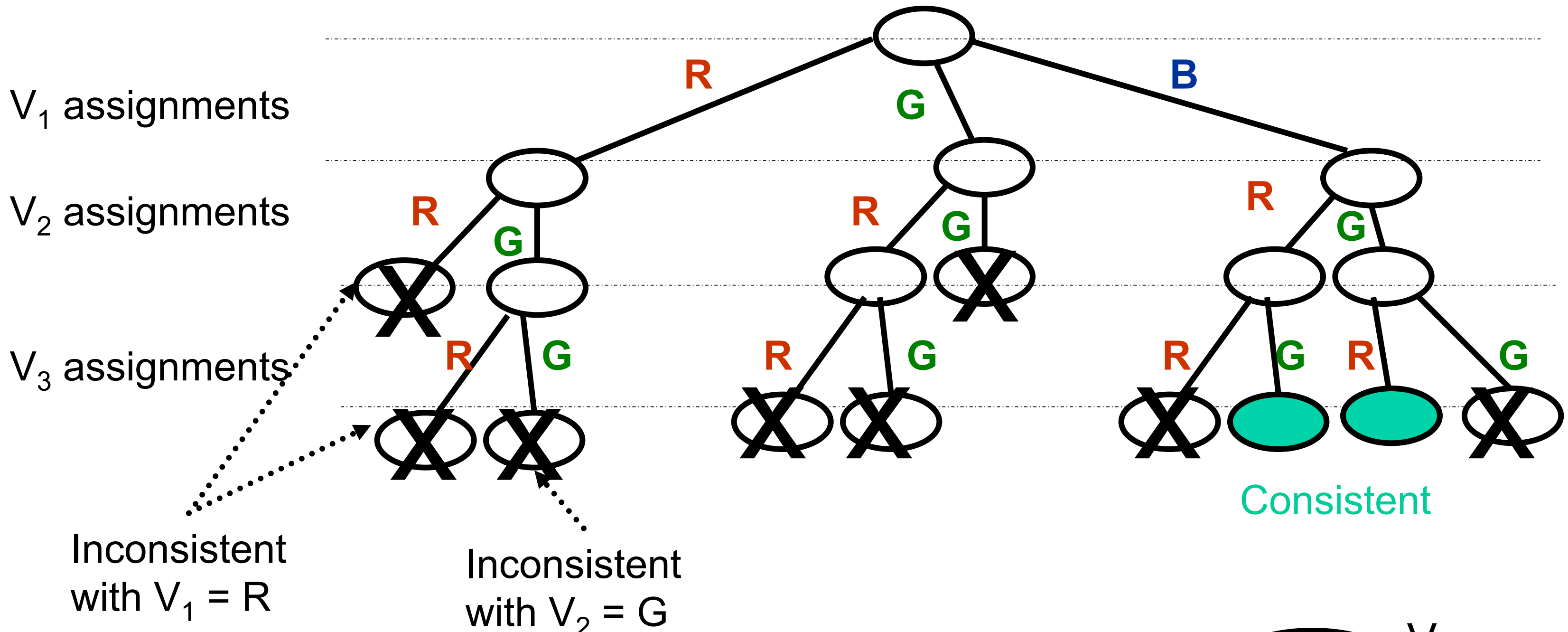
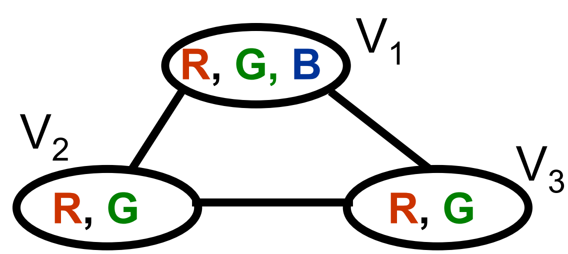


Searching for solutions – backtracking (BT)

When we have too many values in domain (and/or constraints are weak) arc consistency doesn't do much, so we need to search. Simplest approach is pure backtracking (depth-first search).



Backup at inconsistent assignment



Combine Backtracking & Constraint Propagation

A node in BT tree is partial assignment in which the domain of each variable has been set (tentatively) to singleton set.

Use constraint propagation (arc-consistency) to propagate the effect of this tentative assignment, i.e., eliminate values inconsistent with current values.

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Question: How much propagation to do?

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Use constraint propagation (arc-consistency) to propagate the effect of this tentative assignment, i.e., eliminate values inconsistent with current values.

Question: How much propagation to do?

Answer: Not much, just local propagation from domains with unique assignments, which is called forward checking (FC). This conclusion is not necessarily obvious, but it generally holds in practice.

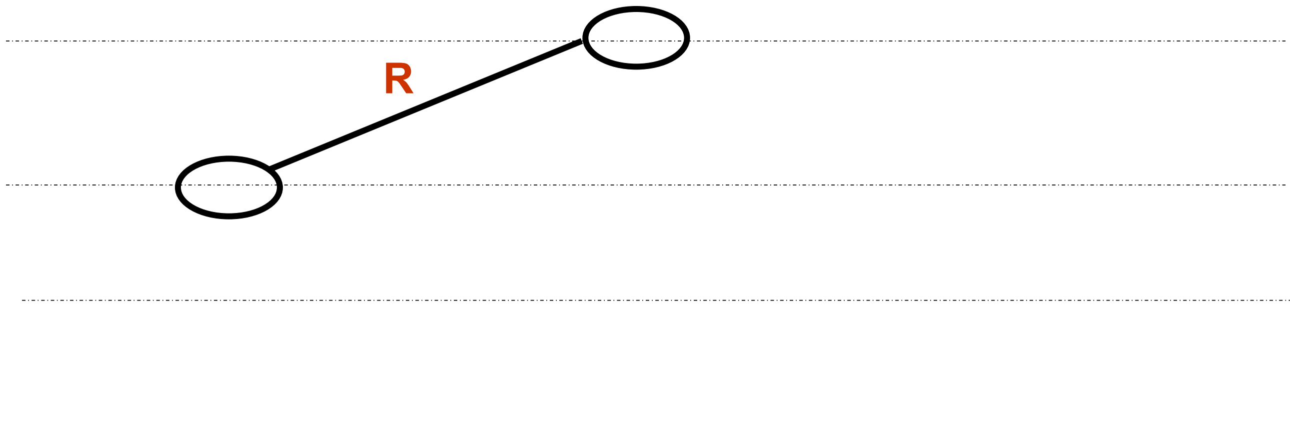
Backtracking with Forward Checking (BT-FC)

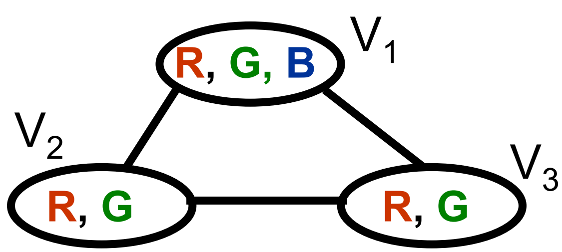
When examining assignment $V_i = d_k$, remove any values inconsistent with that assignment from neighboring domains in constraint graph.

V_1 assignments

V_2 assignments

V_3 assignments





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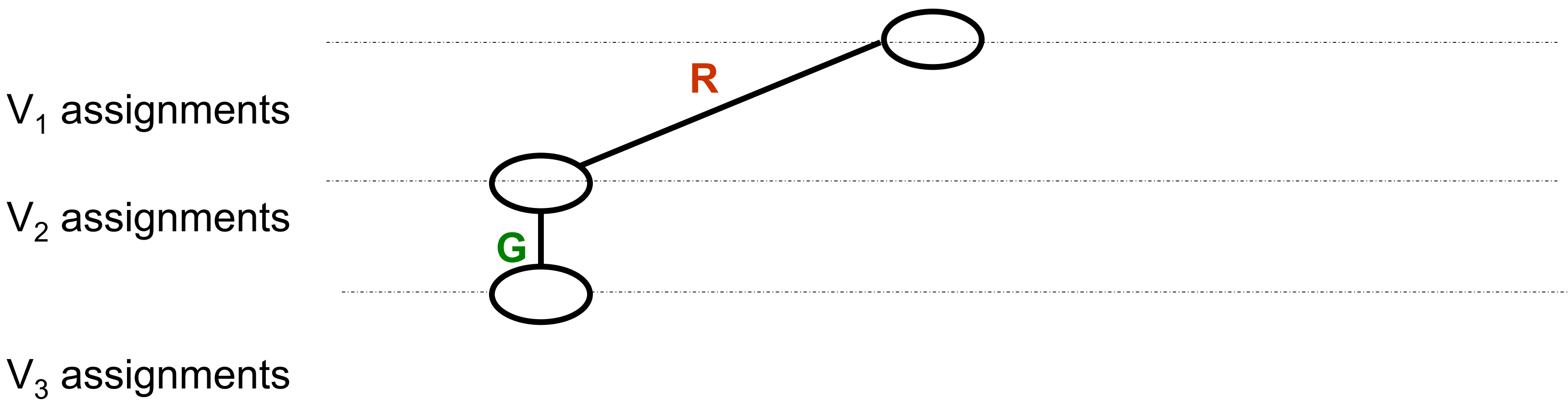

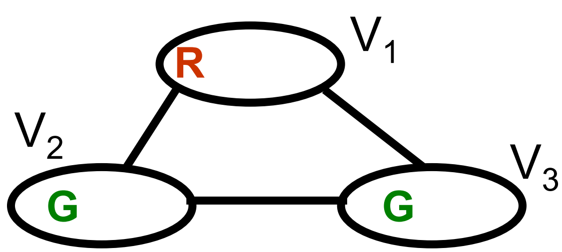

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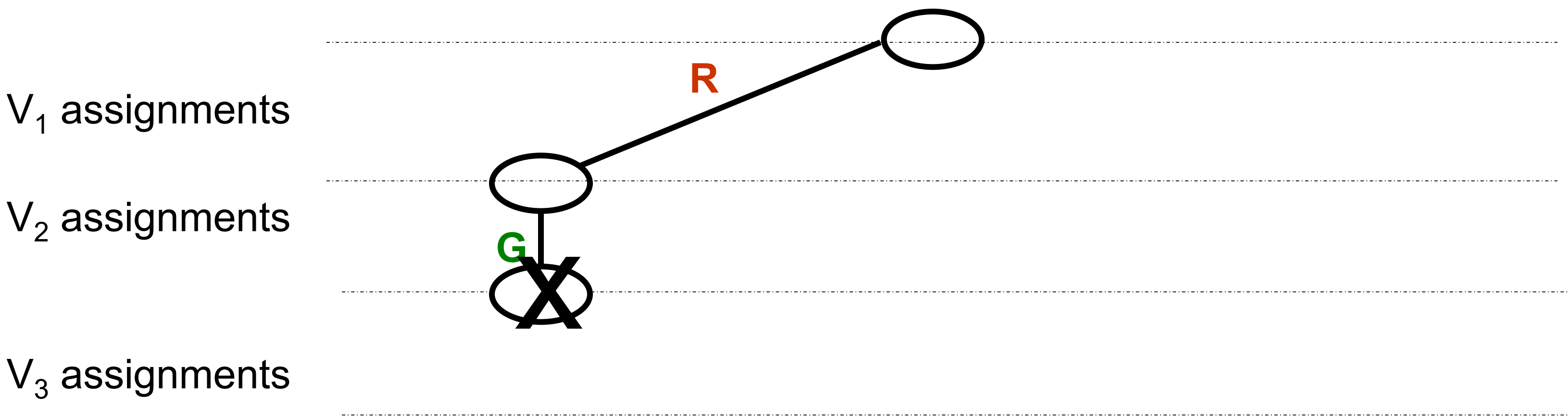
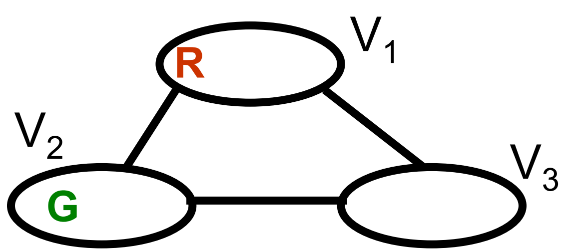


We have a conflict whenever a domain becomes empty.



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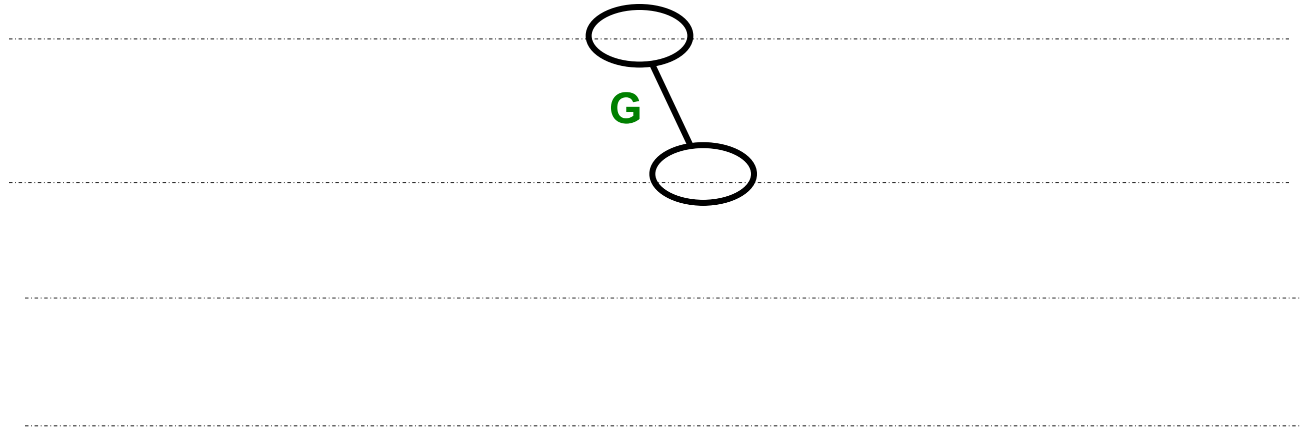
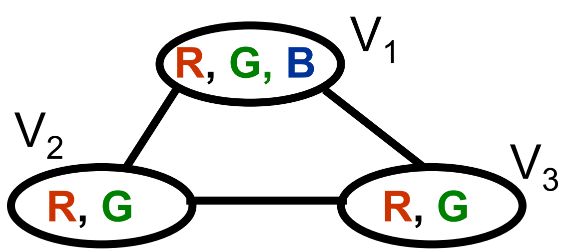


When backing up, need to restore domain values, since deletions were done to reach consistency with tentative assignments considered during search.



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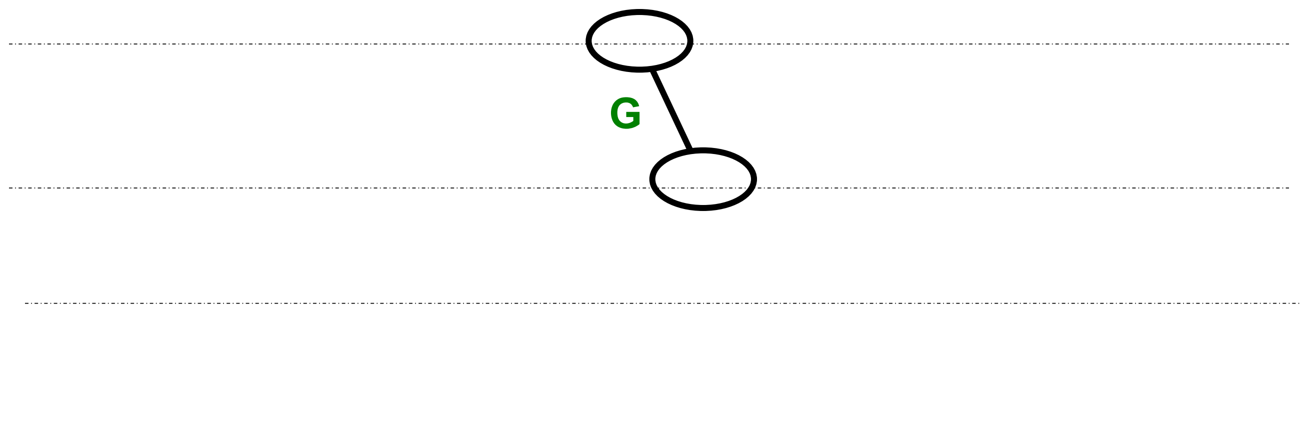

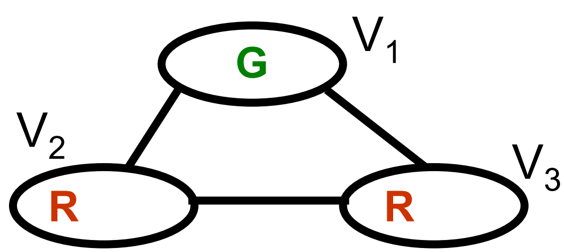

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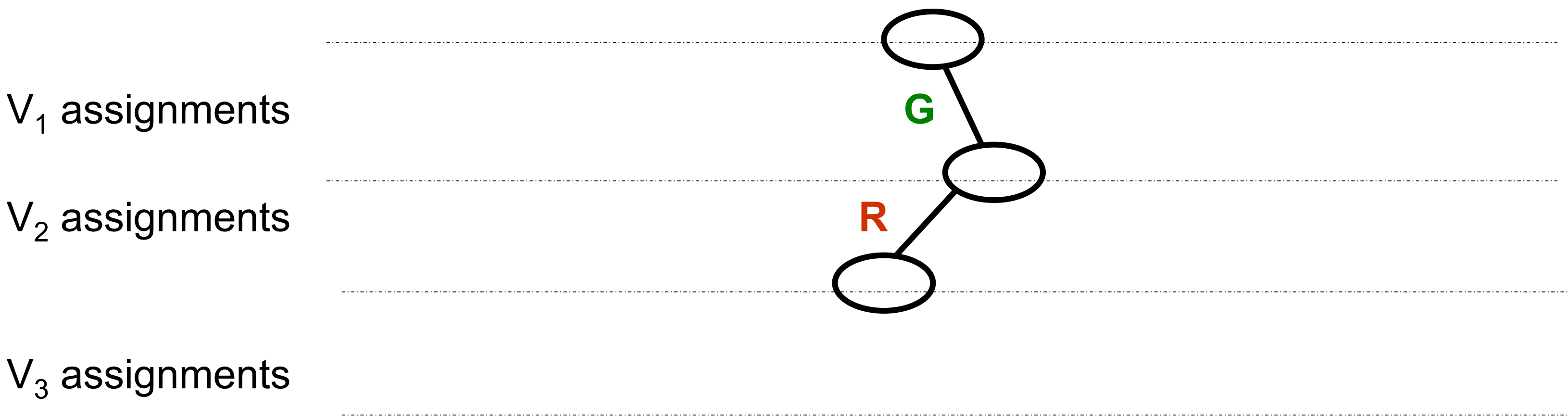

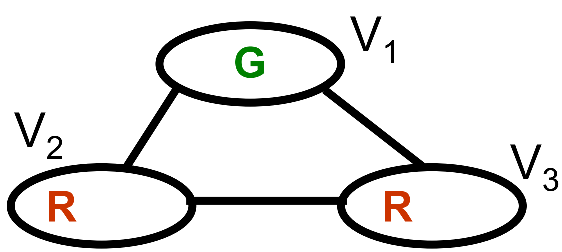

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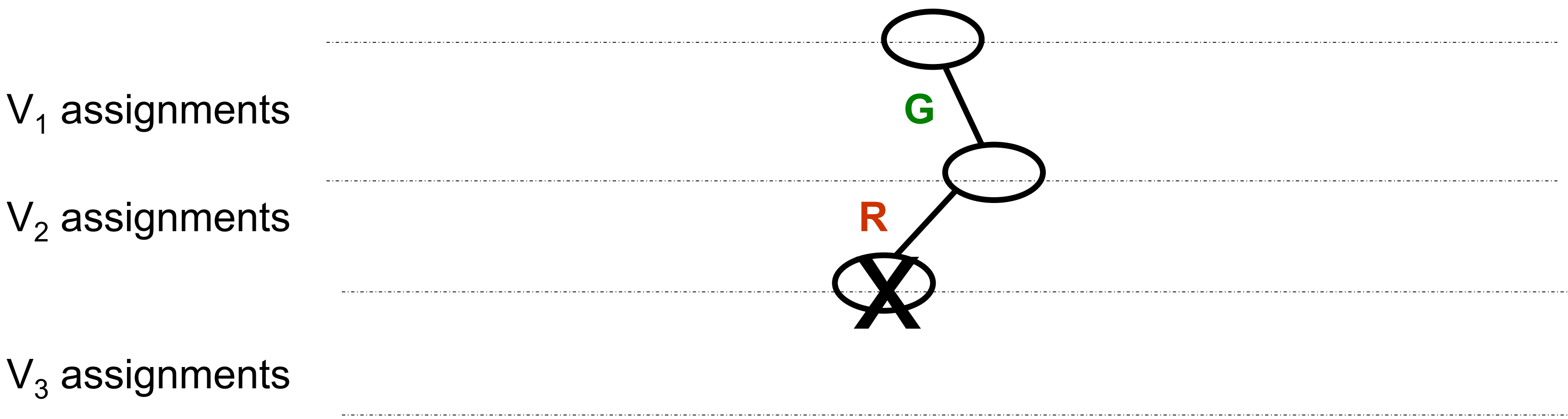

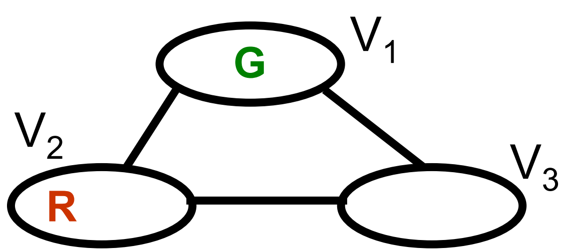

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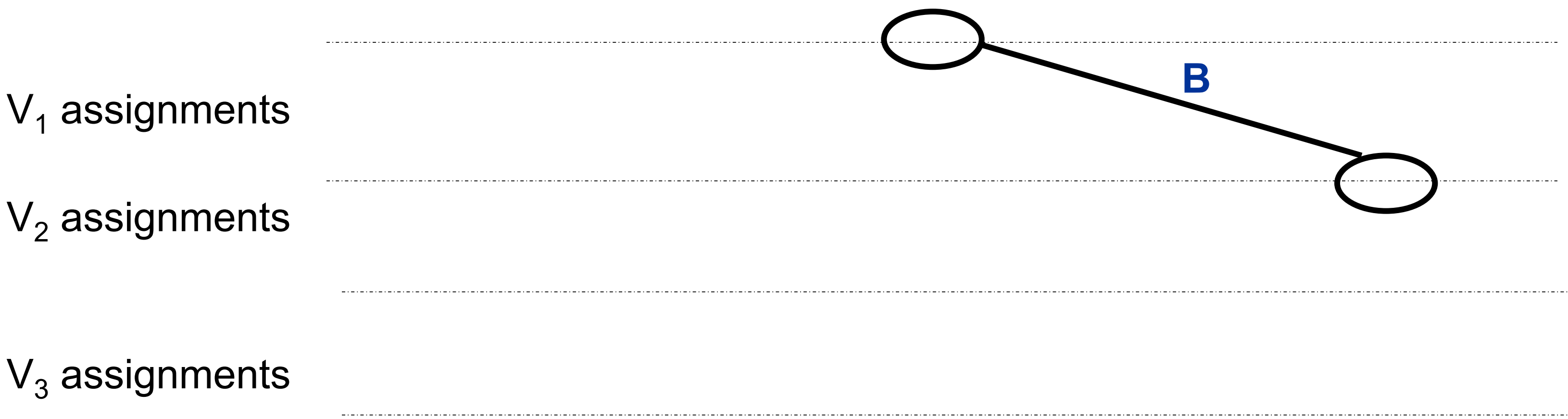

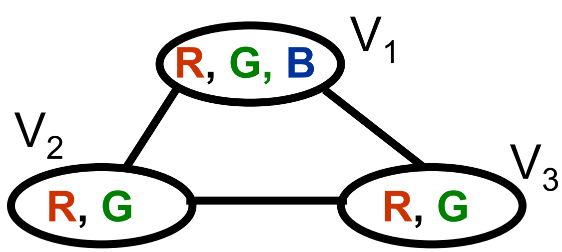

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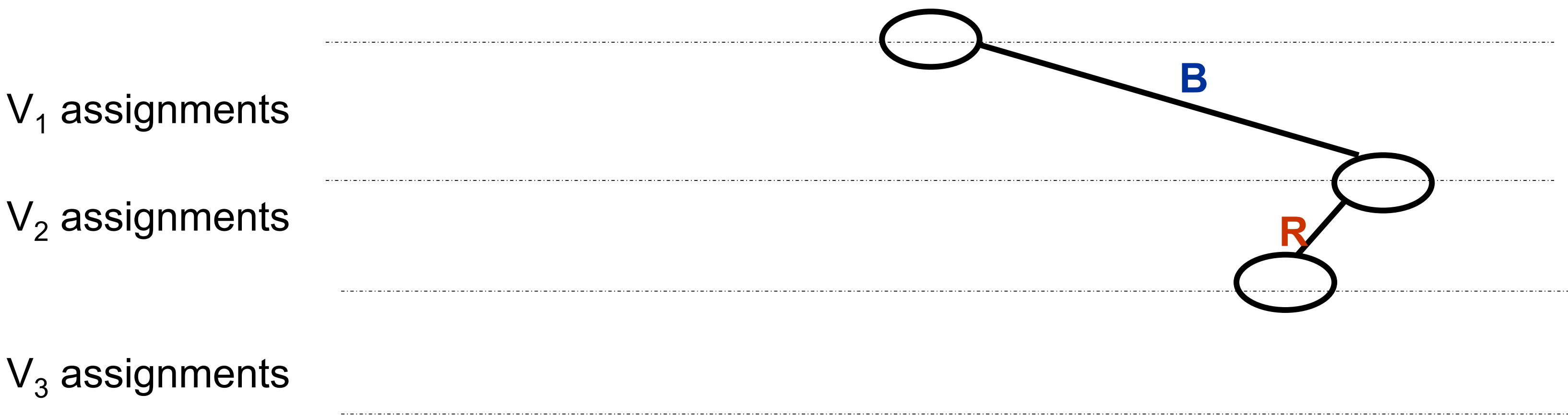

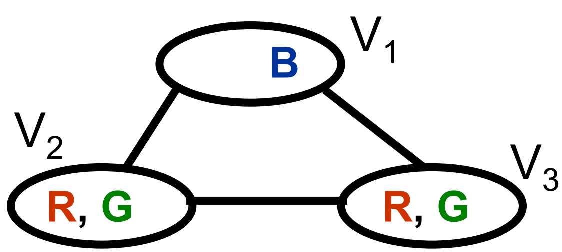

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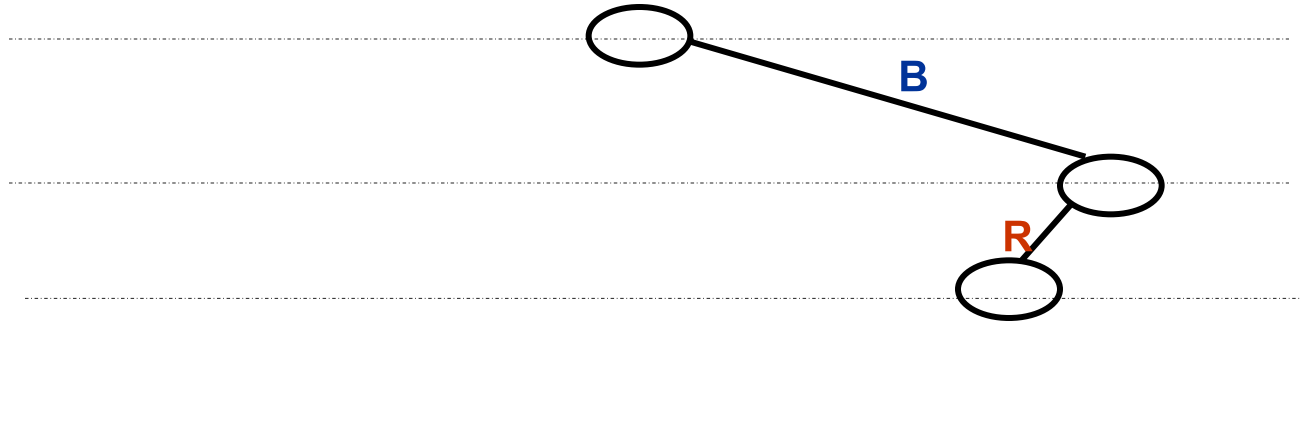

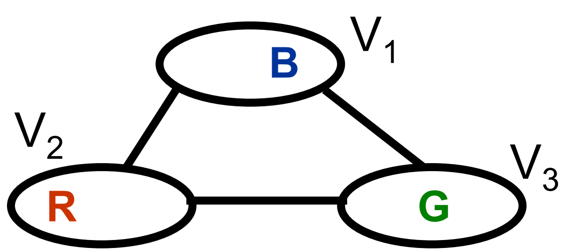

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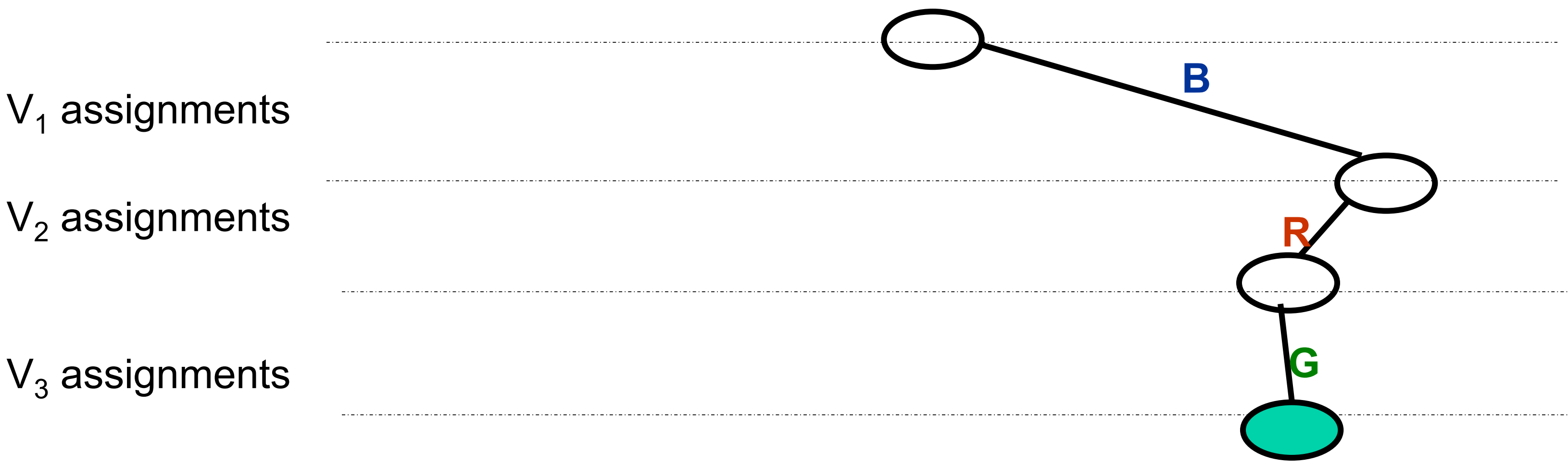

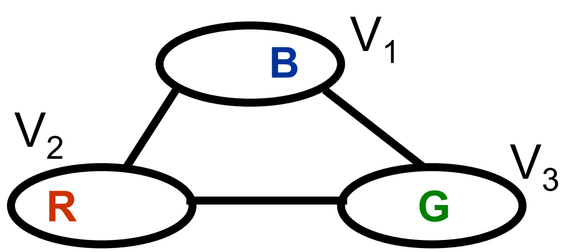

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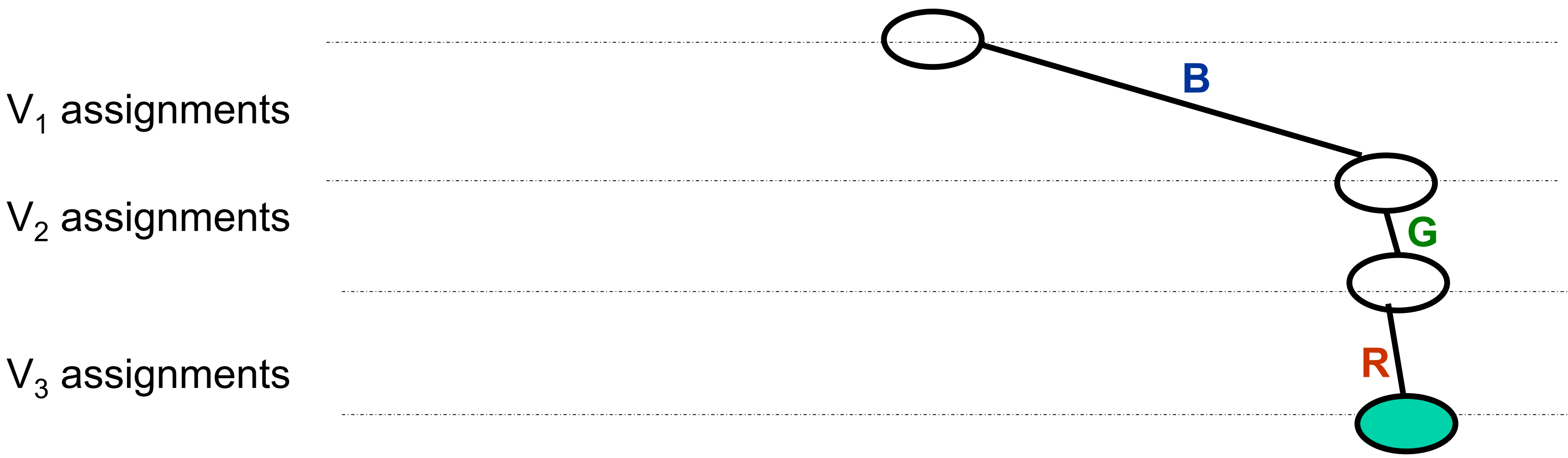

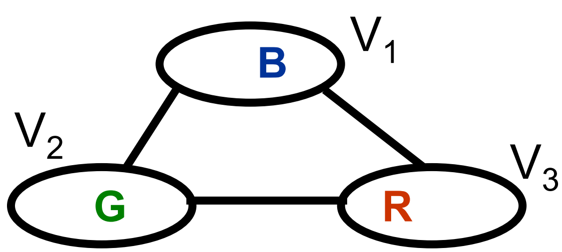

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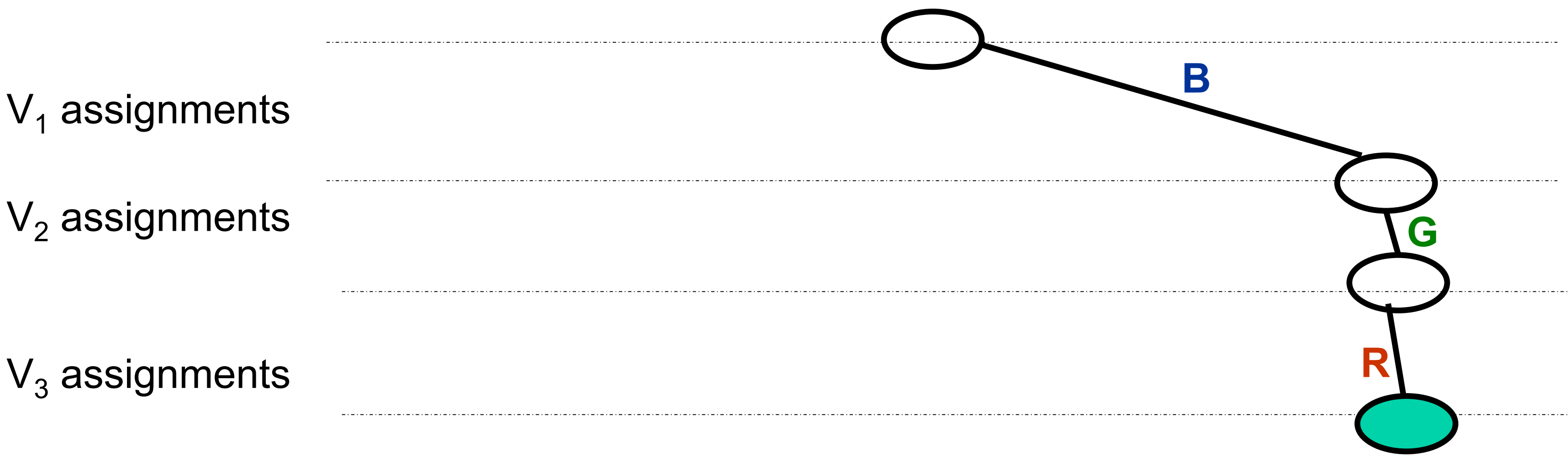
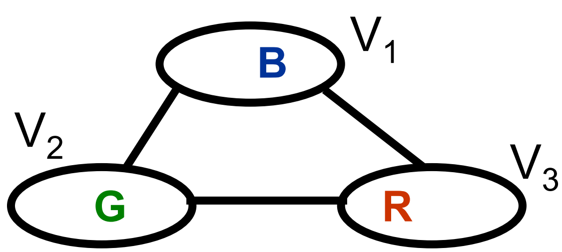


No need to check previous assignments



Generally preferable to pure BT

BT-FC with dynamic ordering

Traditional backtracking uses fixed ordering of variables & values, e.g random order or place variables with many constraints first.

You can usually do better by choosing an order dynamically as the search proceeds.

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  choose value that rules out the fewest values from neighboring domains

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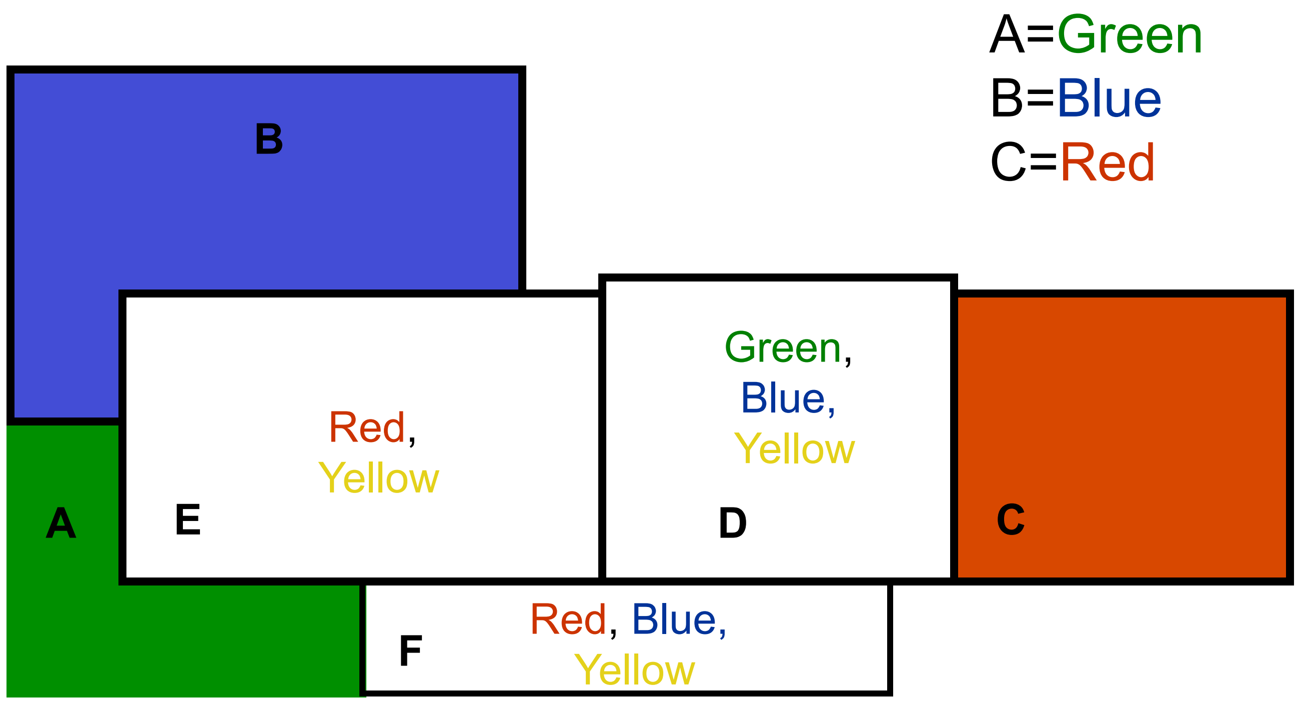
E.g. this combination improves feasible n-queens performance from about n = 30 with just FC to about n = 1000 with FC & ordering.

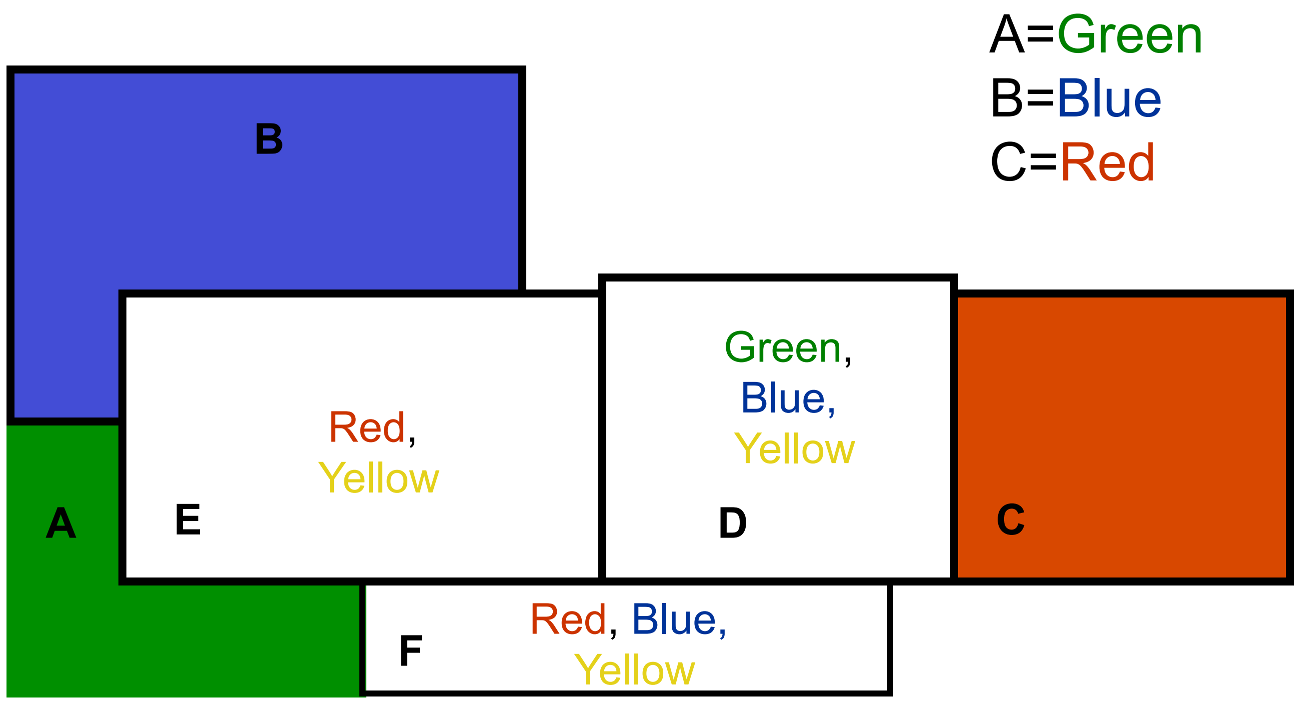
Colors: R, G, B, Y



Which country should we color next →

What color should we pick for it? →

Colors: R, G, B, Y

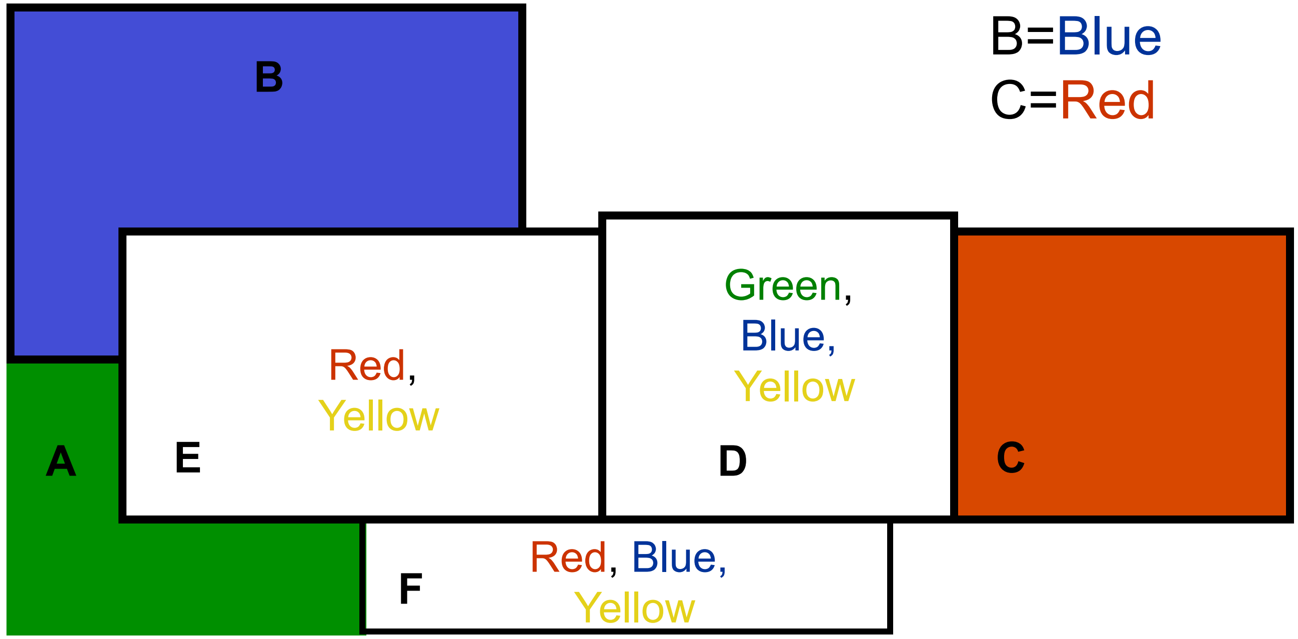


Which country should we color next → E most-constrained variable (smallest domain)

What color should we pick for it? →

Colors: R, G, B, Y

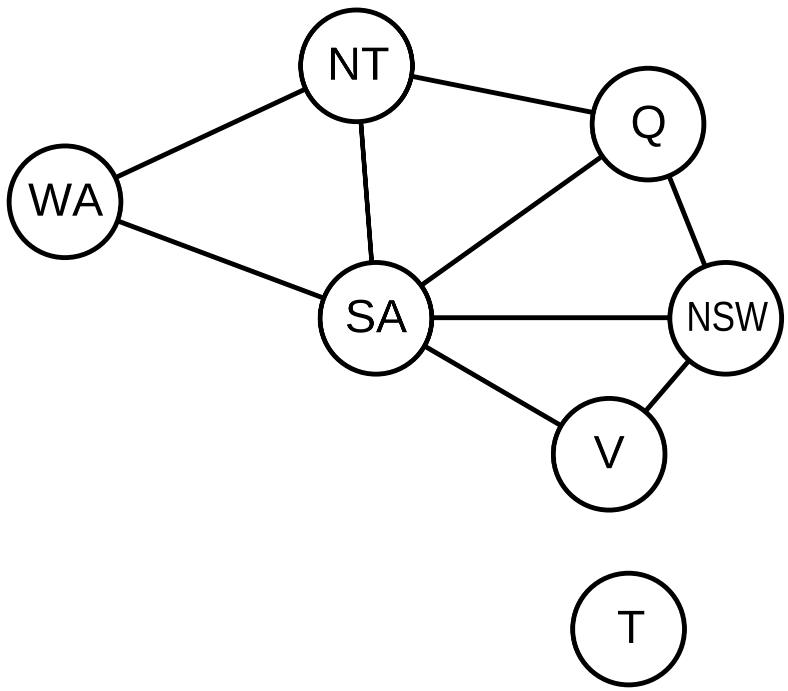
A=Green
B=Blue
C=Red



Which country should we color next → E most-constrained variable (smallest domain)

What color should we pick for it? → RED least-constraining value (eliminates fewest values from neighboring domains)

Problem structure



Tasmania and mainland are independent subproblems

Identifiable as connected components of constraint graph

Problem structure contd.

Suppose each subproblem has c variables out of n total

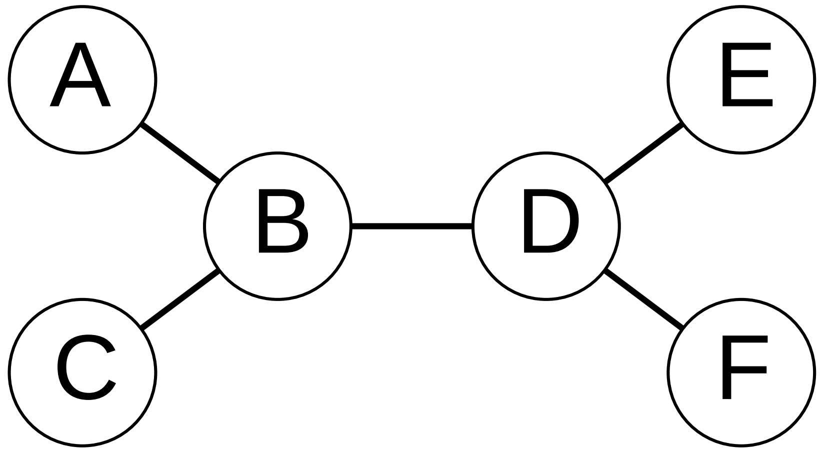
Worst-case solution cost is $n/c \cdot d^c$, **linear** in n

E.g., $n = 80$, $d = 2$, $c = 20$

2^{80} = 4 billion years at 10 million nodes/sec

$4 \cdot 2^{20}$ = 0.4 seconds at 10 million nodes/sec

Tree-structured CSPs



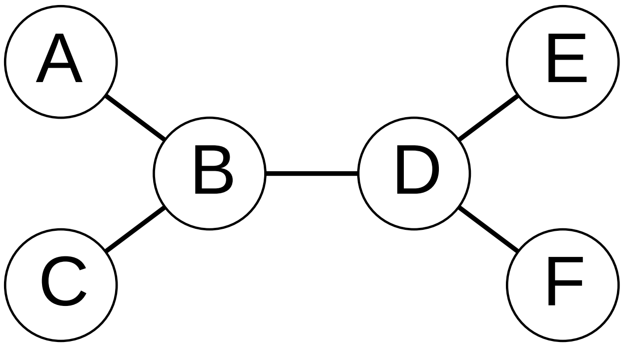
Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

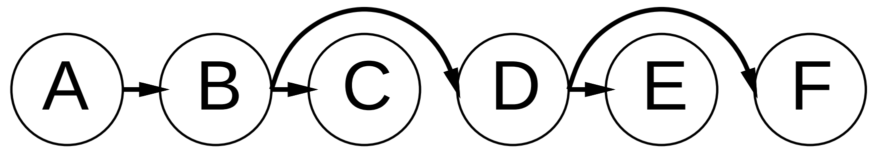
Compare to general CSPs, where worst-case time is $O(d^n)$

This property also applies to logical and probabilistic reasoning:
an important example of the relation between syntactic restrictions
and the complexity of reasoning.

Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves
such that every node's parent precedes it in the ordering



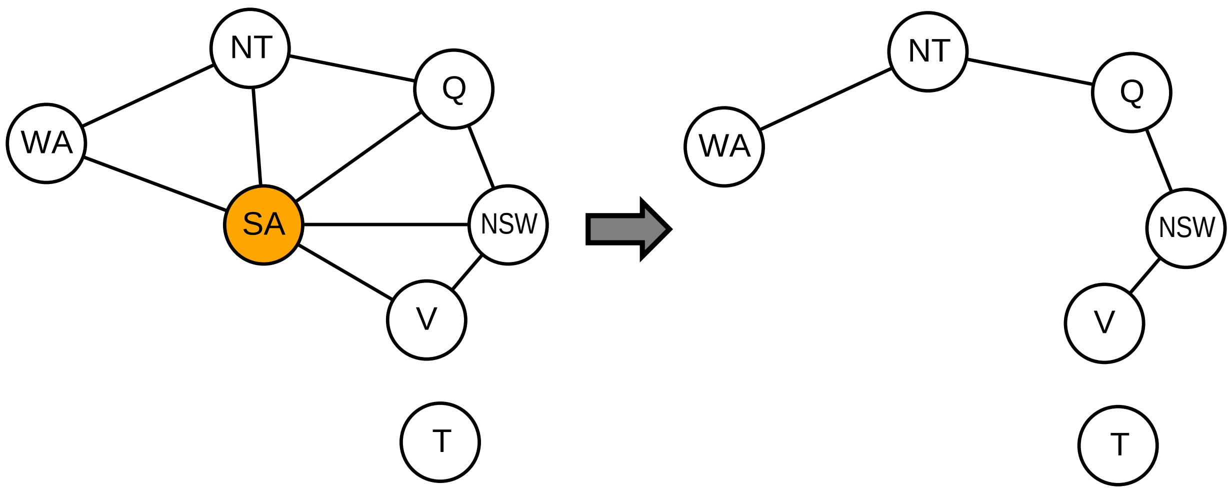


2. For j from n down to 2, apply $\textsc{RemoveInconsistent}(Parent(X_j), X_j)$

3. For j from 1 to n, assign X_j consistently with $Parent(X_j)$

Nearly tree-structured CSPs

Conditioning: instantiate a variable, prune its neighbors' domains



Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

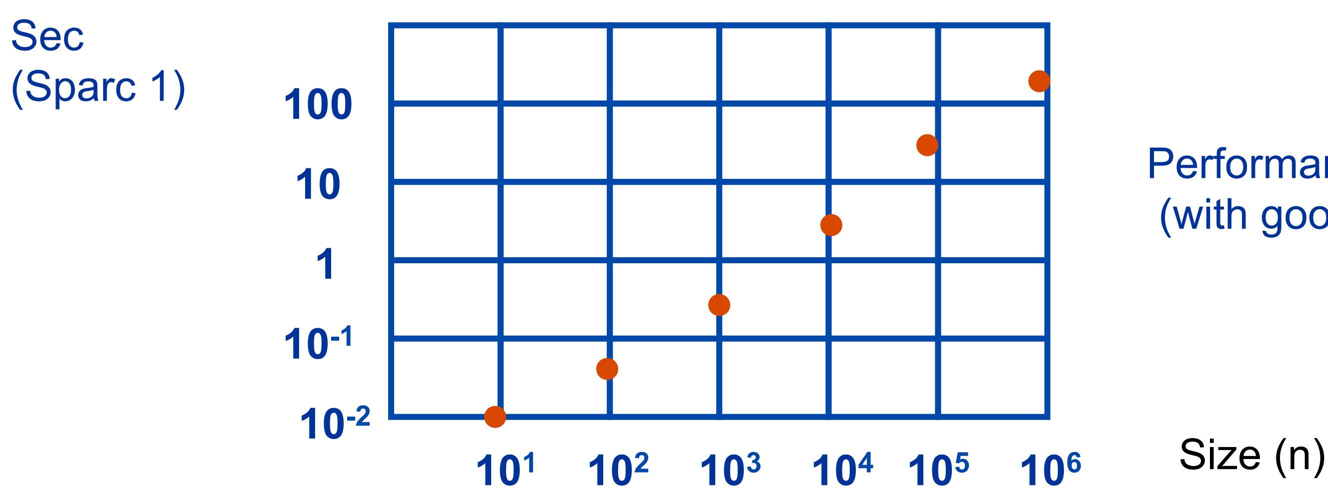
Cutset size $c \Rightarrow$ runtime $O(d^c \cdot (n-c)d^2)$, very fast for small c

Incremental Repair (min-conflict heuristic)

1. Initialize a candidate solution using "greedy" heuristic – get solution "near" correct one.

2. Select a variable in conflict and assign it a value that minimizes the number of conflicts (beak ties randomly).

Can use this heuristic as part of systematic backtracker that uses heuristics to do value ordering or in a local hill-climber (without backup).



Performance on n-queens.
(with good initial guesses)

Min-conflict heuristic

The pure hill climber (without backtracking) can get stuck in local minima. Can add random moves to attempt getting out of minima – generally quite effective. Can also use weights on violated constraints & increase weight every cycle it remains violated.

GSAT

Randomized hill climber used to solve SAT problems. One of the most effective methods ever found for this problem

GSAT as Heuristic Search

- State space: Space of all full assignments to variables
- Initial state: A random full assignment
- Goal state: A satisfying assignment
- Actions: Flip value of one variable in current assignment
- Heuristic: The number of satisfied clauses (constraints); we want to maximize this. Alternatively, minimize the number of unsatisfied clauses (constraints).

GSAT(F)

- For i=1 to Maxtries
  - Select a complete random assignment A
  - Score = number of satisfied clauses
  - For j=1 to Maxflips
    - If (A satisfies all clauses in F) return A
    - Else flip a variable that maximizes score
    - Flip a randomly chosen variable if no variable flip increases the score.

WALKSAT(F)

- For i=1 to Maxtries
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    - If (A satisfies all clauses in F) return A
    - Else
      - With probability p /* GSAT */
        - » flip a variable that maximizes score
        - » Flip a randomly chosen variable if no variable flip increases the score.
      - With probability 1-p /* Random Walk */
        - » Pick a random unsatisfied clause C
        - » Flip a randomly chosen variable in C

Summary

CSPs are a special kind of problem:
    states defined by values of a fixed set of variables
    goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work
to constrain values and detect inconsistencies

The CSP representation allows analysis of problem structure

Tree-structured CSPs can be solved in linear time

Iterative min-conflicts is usually effective in practice