6.034 Quiz 2, Spring 2006
Open Book, Open Notes

1 Propositional Proof (20 pts)

Given the following statements:

1. $P \vee Q$
2. $P \rightarrow R$
3. $Q \rightarrow S$

Prove that $R \vee S$ is entailed, using resolution refutation.

Use the table below. Fill in the Formulas in the proof; in the Reason field, give the parent clause numbers. Start by including the given formulas in the form needed for resolution. You do not need to specify a Reason for the given information. We have given you more than enough space for your proof. You do not need to fill in every line.

Step	Reason	Formula
1	Given	$P \vee Q$
2	Given	$\neg P \vee R$
3	Given	$\neg Q \vee S$
4	Negated Conclusion	$\neg R$
5	Negated Conclusion	$\neg S$
6	2 + 4	$\neg P$
7	6 + 1	Q
8	7 + 3	S
9	8 + 5	False

2 English to FOL (15 points)

Write the following statements in First Order Logic:

1. "Every city has a postman that has been bitten by every dog in the city."
   Use predicates:

   - $City(x)$ means x is a city
   - $Postman(x)$ means x is a postman
   - $Dog(x)$ means x is a dog
   - $Lives(x, y)$ means x lives in city y
   - $Bit(x, y)$ means x bit y

$$\forall c.City(c) \to (\exists p.Postman(p) \land Lives(p, c) \land (\forall d.Dog(d) \land Lives(d, c) \to Bit(d, p)))$$

2. "All blocks supported by blocks that have been moved have also been moved."
   Use predicates:

   - $Block(x)$ means x is a block
   - $Supports(x, y)$ means x supports y
   - $Moved(x)$ means x has been moved

$$\forall x.\forall y.Block(x) \land Block(y) \land Supports(x, y) \land Moved(x) \to Moved(y)$$

3 Logic semantics and interpretation (15 points)

Consider the following interpretation of a language with a unary predicates P, Q, and a binary predicate R.

- Universe $= \{1, 2, 3, 4\}$
- $P = \{\langle 1 \rangle, \langle 3 \rangle\}$
- $Q = \{\langle 2 \rangle, \langle 4 \rangle\}$
- $R = \{\langle 3, 2 \rangle, \langle 4, 3 \rangle, \langle 3, 1 \rangle, \langle 4, 2 \rangle, \langle 2, 1 \rangle, \langle 4, 1 \rangle\}$

Circle the sentences below that hold in that interpretation.

1. $\forall x.P(x)$
2. $\exists x.P(x)$ – Holds
3. $\exists x.P(x) \wedge Q(x)$
4. $\exists x.P(x) \rightarrow Q(x)$ – Holds
5. $\forall x.P(x) \rightarrow Q(x)$
6. $\forall x.P(x) \rightarrow \neg Q(x)$ – Holds
7. $\forall x.Q(x) \rightarrow \neg P(x)$ – Holds
8. $\forall x.\exists y.R(x, y)$
9. $\exists y.\forall x.R(x, y)$
10. $\forall x.P(x) \rightarrow \exists y.R(x, y)$
11. $\forall x.Q(x) \rightarrow \exists y.R(x, y)$ – Holds

6.034 Quiz 3 Solutions, Spring 2006
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1 Clause form and resolution (10 points)

1. Circle the correct clause form for this formula from the choices below.

   $\forall x. \neg(\exists y. P(y) \rightarrow Q(y)) \rightarrow R(x)$

   (a) $P(y) \vee R(x)$
   $\neg Q(y) \vee R(x)$

   (b) $P(f(x)) \vee R(x)$
   $\neg Q(f(x)) \vee R(x)$

   (c) $P(A) \vee R(x)$
   $\neg Q(A) \vee R(x)$

   (d) $\neg P(y) \vee Q(y) \vee R(x)$

   (e) $\neg P(f(x)) \vee Q(f(x)) \vee R(x)$
   - $\forall x. (\exists y. P(y) \rightarrow Q(y)) \vee R(x)$
   - $\forall x. (\exists y. \neg P(y) \vee Q(y)) \vee R(x)$
   - $\forall x. (\neg P(f(x)) \vee Q(f(x))) \vee R(x)$
   - $\neg P(f(x)) \vee Q(f(x)) \vee R(x)$

   (f) $\neg P(A) \vee Q(A) \vee R(x)$

2. Perform resolution on the following clauses; show the unifier and the result.

   $P(f(A), A) \vee \neg Q(f(B), x)$
   $P(f(y), x) \vee Q(x, g(x))$

   (a) Unifier:
   rename variables, so x in first clause is x_1 and x in the second clause is x_2, then $x_1/g(x_2), x_2/f(B)$ or, equivalently, $x_1/g(f(B)), x_2/f(B)$.

   (b) Result: $P(f(A), A) \vee P(f(y), f(B))$

2 Proof (20 points)

This proof encodes the following argument:

- Every integer has at most one predecessor.
- Two is the predecessor of Three.
- The predecessor of Three is even.
- Therefore, Two is even.

To actually carry this out, we also need some axioms about equality, i.e. equality is transitive, equality is symmetric and equals can be substituted in predicates (such as Even).

Fill in any missing Clauses; in the Reason field, give the parent clause numbers, also fill in all the unifiers, written as a set of variable/value bindings.

Step	Reason	Clause	Unifier
1	Given	$\neg Equals(x, y) \vee \neg Equals(z, y) \vee Equals(z, x)$	None
2	Given	$\neg Equals(x, y) \vee Equals(y, x)$	None
3	Given	$\neg Equals(x, y) \vee \neg Even(x) \vee Even(y)$	None
4	Given	$\neg Pred(z, x) \vee Equals(z, Sk1(x))$	None
5	Given	$Pred(Sk0, Three)$	None
6	Given	$Even(Sk0)$	None
7	Given	$Pred(Two, Three)$	None
8	Given	$\neg Even(Two)$	None
9	4,7	$Equals(Two, Sk1(Three))$	$z/Two, x/Three$
10	4,5	$Equals(Sk0, Sk1(Three))$	$z/Sk0, x/Three$
11	1,9	$\neg Equals(z, Sk1(Three)) \vee Equals(z, Two)$	$x/Two, y/Sk1(Three)$
12	10,11	$Equals(Sk0, Two)$	$z/Sk0$
13	3,8	$Equals(x, Two) \vee \neg Even(x)$	y/Two
14	6,13	$\neg Equals(Sk0, Two)$	$x/Sk0$
15	12,14	False	

3 FOL and Entailment (15 points)

Answer each of these questions with **no more than 4 sentences**.

1. Given two first-order logic sentences, A and B, how do you show that A entails B?

   To prove A entails B, we can use FOL Resolution by using A as the KB, introducing $\neg B$ and trying to reach a contradiction. While proof by FOL resolution is sound, it is only semi-decidable; if A does indeed entail B, this will eventually find a contradiction, but if it A does not entail B, it may loop forever.

2. Given two first-order logic sentences, A and B, how do you show that A does not entail B?

   To prove that A does not entail B, we need to find an interpretation for which A is true and B is not. Explicitly searching for such an interpretation is in general intractable (because the number of potential interpretations is infinite and even the size of the interpretation may be infinite).

3. What is it about a domain that would make you want to use first-order logic, rather than propositional logic?

   One situation where FOL is useful is for infinite (or very large) domains; there is no way to express infinite domains in propositional logic. FOL lets you express general statements about relationships among types of objects in the world (or objects based on their properties), rather than merely statements about individual objects themselves.

2. (8 points) For each group of sentences below, give an interpretation that makes the first sentence(s) true and the last sentence false. Use $\{A, B, C\}$ as your universe.

(a)

$$\exists x.p(x) \land q(x) \land r(x,x)$$
$$\forall x.p(x) \rightarrow \exists y.\neg r(x,y)$$
$$\forall x.p(x) \rightarrow \exists y.\neg x = y \land r(x,y)$$

$$\forall x.p(x) \lor \neg q(x)$$

Solution:

$$\begin{aligned} p &= \{< A >\} \\ q &= \{< A, C >\} \\ r &= \{< A, A >, < A, B >\} \end{aligned}$$

(b)

$$\forall x.p(x) \leftrightarrow \exists y.r(y,x)$$
$$\forall x.\exists y.r(x,y)$$

$$\forall x.\neg p(x)$$

Solution:

$$\begin{aligned} p &= \{< A >\} \\ r &= \{< A, A >, < B, A >, < C, A >\} \end{aligned}$$

Alternately: (there are others, too)

$$\begin{aligned} p &= \{< A, B, C >\} \\ r &= \{< A, A >, < B, B >, < C, C >\} \end{aligned}$$

2 FOL Semantics

(**6**) Consider a world with objects **A**, **B**, and **C**. We'll look at a logical languge with constant symbols X, Y, and Z, function symbols f and g, and predicate symbols p, q, and r. Consider the following interpretation:

- $I(X) = \mathbf{A}$, $I(Y) = \mathbf{A}$, $I(Z) = \mathbf{B}$
- $I(f) = \{\langle \mathbf{A}, \mathbf{B} \rangle, \langle \mathbf{B}, \mathbf{C} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$
- $I(p) = \{\mathbf{A}, \mathbf{B}\}$

- $I(q) = \{\mathbf{C}\}$
- $I(r) = \{\langle\mathbf{B},\mathbf{A}\rangle, \langle\mathbf{C},\mathbf{B}\rangle, \langle\mathbf{C},\mathbf{C}\rangle\}$

For each of the following sentences, say whether it is true or false in the given interpretation I:

a. $q(f(Z))$

b. $r(X, Y)$

c. $\exists w.f(w) = Y$

d. $\forall w.r(f(w), w)$

e. $\forall u, v.r(u, v) \to (\forall w.r(u, w) \to v = w)$

f. $\forall u, v.r(u, v) \to (\forall w.r(w, v) \to u = w)$

3 Interpretations

(**6**) Using the same set of symbols as in the previous problem, for each group of sentences below, provide an interpretation that makes the sentences true, or show that it's impossible.

a.
- $\exists w.p(w) \wedge \exists w.q(w)$
- $\neg\exists w.p(w) \wedge q(w)$
- $\forall u.p(u) \to \exists v.r(u, v)$

b.
- $\forall u.\exists v.r(u, v)$
- $\exists u, v.\neg r(u, v)$
- $\forall v.(\exists u.r(u, v)) \leftrightarrow p(v))$

c.
- $\forall u, v.(p(v) \to r(u, v))$
- $\exists u, v.\neg r(u, v)$
- $\exists v.p(v)$

2 FOL Semantics

a. T

b. F

c. F

d. T

e. F

f. T

3 Interpretations

a.
- $I(p) = \{\mathbf{A}\}$
- $I(q) = \{\mathbf{C}\}$
- $I(r) = \{\langle \mathbf{A}, \mathbf{B} \rangle\}$

b.
- $I(p) = \{\mathbf{B}, \mathbf{C}\}$
- $I(r) = \{\langle \mathbf{A}, \mathbf{B} \rangle, \langle \mathbf{B}, \mathbf{B} \rangle, \langle \mathbf{C}, \mathbf{C} \rangle\}$

c.
- $I(p) = \{\mathbf{A}\}$
- $I(r) = \{\langle \mathbf{A}, \mathbf{A} \rangle, \langle \mathbf{B}, \mathbf{A} \rangle, \langle \mathbf{C}, \mathbf{A} \rangle\}$

2. **First-order logic**
   Consider the following sentences:

   - If a jar is sterile, then there are no live bacteria in it.
   - There are live bacteria in the yogurt cup.
   - The yogurt cup is a jar.
   - The yogurt cup is not sterile.
   - If there are no live bacteria in a jar, then it is sterile.

   (a) Write each of these sentences in first-order logic, using predicates *Jar*, *Sterile*, *Bacterium*, *Live*, and *In*, and the constant symbol *YogurtCup*.

Answer:

$$\forall x. Jar(x) \land Sterile(x) \Rightarrow \neg \exists y. Bacterium(y) \land In(y, x) \land Live(y) \tag{1}$$

$$\exists x. Bacterium(x) \land In(x, YogurtCup) \land Live(x) \tag{2}$$

$$Jar(YogurtCup) \tag{3}$$

$$\neg Sterile(YogurtCup) \tag{4}$$

$$\forall x. Jar(x) \land (\neg \exists y. Live(y) \land Bacterium(y) \land In(y, x)) \Rightarrow Sterile(x) \tag{5}$$

(c) How can you show that a set of sentences entails another sentence?

Answer: You can show that $KB \models S$ by showing that $KB \vdash S$: negate the sentence S and show using FOL inference rules (resolution, paramodulation) that $KB \wedge \neg S$ leads to a contradiction (empty clause). This shows that the set of interpretations under which KB holds is a subset of the set of interpretations under which S holds.

(d) How can you show that a set of sentences does not entail another sentence?

Answer: You can show that $KB \not\models S$ by showing the existence of an interpretation under which KB holds but S does not. Showing that resolution-refutation cannot reach a contradiction does not work in general because if $KB \not\models S$, resolution-refutation may never terminate due to semi-decidability of FOL.

(e) Do the first three sentences entail the fifth? Show your answer using one of the two methods you just described.

Answer: No, they do not. Consider the universe $U = \{YogurtCup, Keg, SpongeBob\}$ and the interpretation:

$$
\begin{aligned}
I(jar) &= \{< YogurtCup >, < Keg >\} \\
I(Bacterium) &= \{< SpongeBob >\} \\
I(In) &= \{< SpongeBob, YogurtCup >\} \\
I(Live) &= \{< SpongeBob >\} \\
I(Sterile) &= \{\}
\end{aligned}
$$

Under this interpretation, (1) through (3) hold, but (5) does not (Keg is a jar with no live bacteria in it, and it is not sterile).

3 The logic of file systems (20 points)

In this problem, we'll formalize the permissions of files and directories in a file system. Here are the intended interpretations of the predicates we'll use:

- $Owns(x, y)$: person x owns object y
- $In(x, y)$: object x (which could be a file or a directory) is in directory y (note that by *in* we mean a child, not a descendant)
- $File(x)$: x is a file
- $Dir(x)$: x is a directory
- $Ne(x)$: x is a non-empty directory
- J: constant standing for Joan

Here are some sentences in first-order logic:

1. $\forall x, y, z. Owns(x, y) \wedge In(z, y) \rightarrow Owns(x, z)$
2. $\forall x. Ne(x) \leftrightarrow \exists y. In(y, x) \wedge File(y)$
3. $\exists x. Owns(J, x) \wedge Ne(x)$
4. $\exists x. Owns(J, x) \wedge File(x)$

3.1 Proof

Converting the first three of these sentences and the negation of the fourth into clausal form, we get the clauses on the top of the next page. Use resolution refutation to derive a contradiction, demonstrating that the fourth sentence is entailed by the first three. It is not necessary to fill in every line of the proof table.

1. $\neg Owns(x_1, y_1) \vee \neg In(z_1, y_1) \vee Owns(x_1, z_1)$
2. $\neg Ne(x_2) \vee In(f(x_2), x_2)$
3. $\neg Ne(x_3) \vee File(f(x_3))$
4. $\neg In(y_4, x_4) \vee \neg File(y_4) \vee Ne(x_4)$
5. $Owns(J, A)$
6. $Ne(A)$
7. $\neg Owns(J, x_7) \vee \neg File(x_7)$

Step	P1	P2	Clause	Unifier
8	2	6	$In(f(A), A)$	$\{x_2/A\}$
9	1	5	$\neg In(z_1, A) \vee Owns(J, z_1)$	$\{x_1/J, y_1/A\}$
10	8	9	$Owns(J, f(A))$	$\{z_1/f(A)\}$
11	3	6	$File(f(A))$	$\{x_3/A\}$
12	7	11	$\neg Owns(J, f(A))$	$x_7/f(A)$
13	10	12	*false*	
14				
15				
16				

6.034 Quiz 2 Answers, Spring 2004

1 First-Order Logic (24 points)

Here are some English sentences and their translation into clausal form.

1. Every car has a driver.
$$D(f(x_1), x_1)$$
2. The driver of a car is in the car.
$$\neg D(x_2, y_2) \vee In(x_2, y_2)$$
3. "In" is transitive.
$$\neg In(x_3, y_3) \vee \neg In(y_3, z_3) \vee In(x_3, z_3)$$
4. Drivers are people.
$$\neg D(x_4, y_4) \vee P(x_4)$$
5. Chitty (a car) is in the Stata garage.
$$In(C, SG)$$
6. Therefore, there is a person in the Stata garage. (This clause is the negation of the conclusion).
$$\neg P(x_6) \vee \neg In(x_6, SG)$$

We'd like to prove the conclusion using resolution refutation. This proof is kind of tricky, so we're going to tell you, in English, what the steps should be. For each step, say which of the previous clauses (P1 and P2 in the table) it can be derived from using resolution, what the resulting clause is and what the unifier is.

Step	P1	P2	Clause	Unifier
7	Every driver is in their car. (one term)			
	1	2	$In(f(x_1), x_1)$	$\{x_2/f(x_1), y_2/x_1\}$
8	If a car is in some location, then its driver is in that location. (two terms)			
	3	7	$\neg In(x_1, z_3) \vee In(f(x_1), z_3)$	$\{x_3/f(x_1), y_3/x_1\}$
9	The driver of a car is a person. (one term)			
	1	4	$P(f(x_1))$	$\{x_4/f(x_1), y_4/x_1\}$
10	The driver of Chitty is in the Stata garage. (one term)			
	5	8	$In(f(C), SG)$	$\{x_1/C, z_3/SG\}$
11	There is no car whose driver is in the Stata garage. (one term)			
	6	9	$\neg In(f(x_1), SG)$	$\{x_6/f(x_1)\}$
12	False			
	10	11	Nil	$\{x_1/C\}$

9 Resolution Proof (15 points)

Prove a contradiction from these clauses using resolution. For each new step, indicate which steps it was derived from (in columns labeled P1 and P2) and what the unifier was. Note that A is a constant and b, c, d, x, y, u, v, w are all variables.

This is just an example answer; there were lots of orders in which this could be done.

Step	P1	P2	Clause	Unifier
1	XX	XX	$\neg P(x, f(x), y) \vee R(y, g(x))$	XXXXXXXXX
2	XX	XX	$\neg R(u, v) \vee \neg Q(v) \vee S(u, h(v))$	XXXXXXXXX
3	XX	XX	$Q(g(A))$	XXXXXXXXX
4	XX	XX	$\neg S(w, w)$	XXXXXXXXX
5	XX	XX	$P(b, c, h(d))$	XXXXXXXXX
6	1	5	$R(h(d), g(b))$	$\{x/b, c/f(x), y/h(d)\}$
7	2	6	$\neg Q(g(b)) \vee S(h(d), h(g(b)))$	$\{u/h(d), v/g(b)\}$
8	4	7	$\neg Q(g(b))$	$\{w/h(d), d/g(b)\}$
9	3	8	*False*	$\{b/A\}$
10				

Given the following clauses, do a resolution refutation proof. *(10 points)*

1. ¬P(x,f(x))∨¬R(f(x))∨¬Q(x,g(x))
2. ¬P(x2,y2)∨ Q(x2,y2)
3. ¬P(x3,y3)∨ R(y3)
4. P(A,x4)   [Negated Goal]

Step	Parent	Parent	New Clause	MGU
5	*3*	*4*	*R(y3)*	*x3=A, y3=x4*
6	*2*	*4*	*Q(A, y2)*	*x2=A, y2=x4*
7	*1*	*5*	*¬P(x,f(x))∨¬Q(x,g(x))*	*y3=f(x)*
8	*6*	*7*	*¬P(A,f(A))*	*x=A, y2=g(x)*
9	*4*	*8*	*( )*	*x4=f(A)*

or

Step	Parent	Parent	New Clause	MGU
5	*1*	*4*	*¬R(f(A)) ∨¬Q(A,g(A))*	*x=A, x4=f(x)*
6	*3*	*5*	*¬P(x3,f(A)) ∨¬Q(A,g(A))*	*y3=f(A)*
7	*2*	*6*	*¬P(x3,f(A)) ∨¬P(A,g(A))*	*x2=A, y2=g(A)*
8	*4*	*7*	*¬P(A,g(A))*	*x3=A, x4=f(A)*
9	*4*	*8*	*( )*	*x4=g(A)*

There are other possibilities as well. Note that (whenever possible) you want to use the shortest clauses in the resolution steps.

C. Given the following clauses:

1. Hasjob(p, job(p))
2. ¬ Hasjob(p, k) ∨ Equal(job(p), k)
3. Hasjob(George, Fireman)
4. ¬ Equal(Fireman, Teacher)
5. ¬ Equal(x,y) ∨ ¬ Equal(y, z) ∨ Equal(x, z)
6. ¬ Equal(x,y) ∨ Equal(y,x)

Prove by resolution refutation that:
¬ Hasjob(George, Teacher)

Hint: think about the strategy for the proof before you start doing resolutions. How would you prove the result by hand?

Step	**Parent**	**Parent**		**Unifier**
7	Neg	Goal	Hasjob(George, Teacher)	-----------------
8	2	7	Equal(job(George), Teacher)	p=George k=Teacher
9	2	3	Equal(job(George), Fireman)	p=George k=Fireman
10	9	6	Equal(Fireman, job(George))	x=job(George) y=Fireman
11	5	10	¬ Equal(job(George),z) ∨ Equal(Fireman, z)	x=Fireman y=job(George)
12	8	11	Equal(Fireman,Teacher)	z=Teacher
13	4	12	Contradiction	
14				