MASSACHVSETTS INSTITVTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science
6.01—Introduction to EECS I
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Lecture 8

Circuit Equivalents

Circuit Equivalents

We saw last week that pieces of circuits cannot be abstracted as functional elements; the actual voltages and currents in them will depend on how they are connected to the rest of a larger circuit. However, we can still abstract them as sets of constraints on the values involved.

In fact, when a circuit includes only resistors and voltage sources, we can derive a much simpler circuit that induces the same constraints on currents and voltages as the original one. This a kind of abstraction that's similar to the abstraction that we saw in linear systems: we can take a complex circuit and treat it as if it were a much simpler circuit.

If somebody gave you a circuit made of resistors and voltage sources, and put it in a black box with two wires coming out, labeled + and -, what could you do with it? You could try to figure out what constraints that box puts on the voltage between and current through the wires coming out of the box.

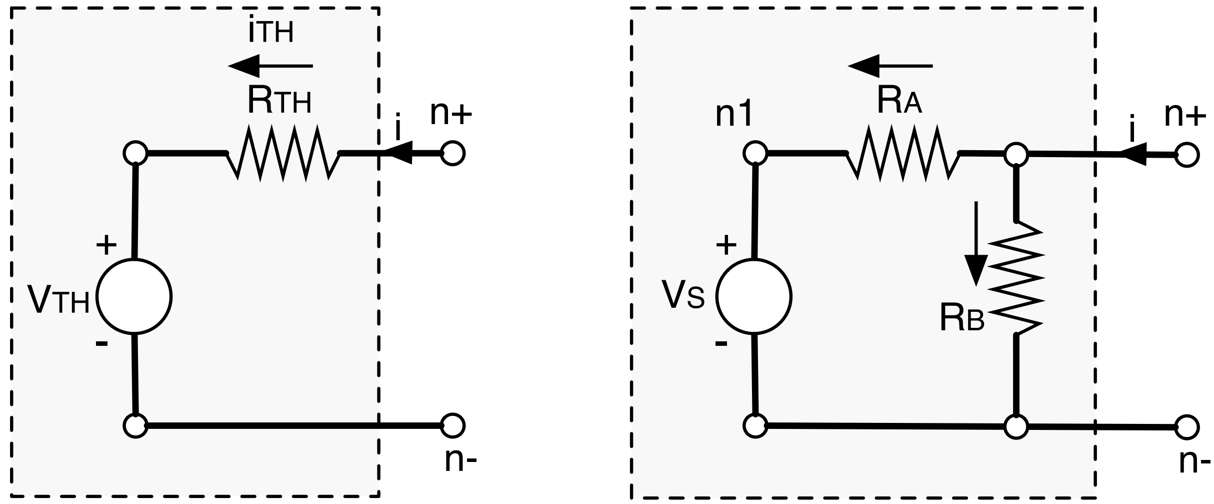
We can start by figuring out the *open-current voltage* across the two terminals. That is the voltage drop we'd see across the two wires if nothing were connected to them. We'll call that V_{oc}. Another thing we could do is connect the two wires together, and see how much current runs through them; this is called the *short-circuit current*. We'll call that i_{sc}.

It turns out that these two values are sufficient to characterize the constraint that this whole box will exert on a circuit connected to it. The constraint will be a relationship between the voltage across its terminals and the current flowing through the box. We can derive it by using Thévenin's theorem:

Theorem 1 *Any combination of voltage sources and resistances with two terminals can be replaced by a single voltage source* V_{th} *and a single series resistor* R_{th}*. The value of* V_{th} *is the open circuit voltage at the terminals* V_{oc}*, and the value of* R_{th} *is* V_{th} *divided by the current with the terminals short circuited* ($-i_{sc}$)*.*

Let's look at a picture, then an example. In figure 1(a) we show a picture of a black (well, gray) box, abstracted as being made up of a circuit with a single voltage source V_{th} and a single resistor R_{th} in series. The open-circuit voltage from n_+ to n_- is clearly V_{th}. The short-circuit current i_{sc} (in the direction of the arrow) is $-V_{th}/R_{th}$. So, this circuit would have the desired measured properties.[1]

[1] The minus sign here can be kind of confusing. The issue is this: when we are treating this circuit as a black box with terminals n_+ and n_-, we think of the current flowing *out* of n_+ and *in* to n_-, which is consistent with the voltage difference $V_{th} = V_+ - V_-$. But when we compute the short-circuit current by wiring n_+ and n_- together, we are continuing to think of i_{sc} as flowing out of n_+, but now it is coming *out* of n_- and *in* to n_+, which is the opposite direction. So, we have to change its sign to compute R_{th}.



(a) The abstract Thévenin equivalent circuit

(b) An example circuit

Figure 1: Thévenin equivalence examples

Figure 1(b) shows an actual circuit. We'll compute its associated open-circuit voltage and short-circuit current, construct the associated *Thévenin equivalent* circuit, and be sure it has the same properties.

The first step is to compute the open-circuit voltage. This just means figuring out the difference between the voltage at nodes n_+ and n_-, under the assumption that the current $i = 0$. An easy way to do this is to set n_- as ground and then find the node voltage at n_+. Let's write down the equations:

$$\begin{aligned}
v_+ - v_1 &= i_A R_A \\
v_1 - v_- &= V_s \\
v_+ - v_- &= i_B R_B \\
-i_A - i_B &= 0 \\
i_A - i_S &= 0 \\
v_- &= 0
\end{aligned}$$

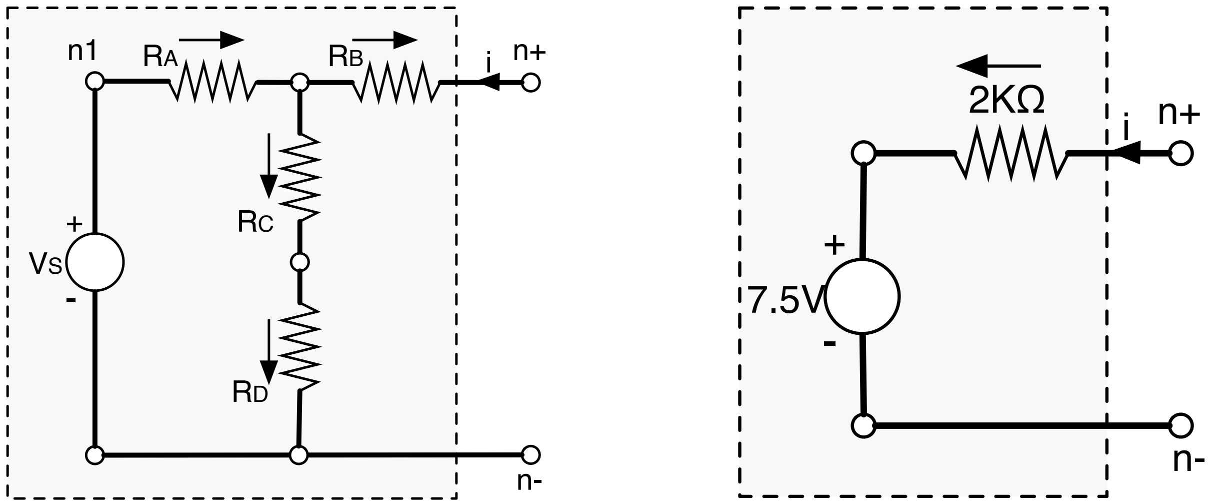
We can solve these pretty straightforwardly to find that

$$v_+ = V_s \frac{R_B}{R_A + R_B} \quad .$$

So, we know that, for this circuit, $R_{th} = V_s \frac{R_B}{R_A + R_B}$.

Now, we need the short-circuit current, i_{sc}. To find this, imagine a wire connecting n_+ to n_-; we want to solve for the current passing through this wire. We can use the equations we had before, but adding equation 4 wiring n_+ to n_-, and adding the current i_{sc} to the KCL equation 5.

$$v_+ - v_1 = i_A R_A \tag{1}$$
$$v_1 - v_- = V_s \tag{2}$$



(a) Another example circuit

(b) Its Thévenin equivalent circuit

Figure 2: Thévenin equivalence examples

$$v_+ - v_- = i_B R_B \quad (3)$$
$$v_+ = v_- \quad (4)$$
$$i_{sc} - i_A - i_B = 0 \quad (5)$$
$$i_A - i_S = 0 \quad (6)$$
$$v_- = 0 \quad (7)$$

We can solve this system to find that

$$i_{sc} = -\frac{V_s}{R_A} \;,$$

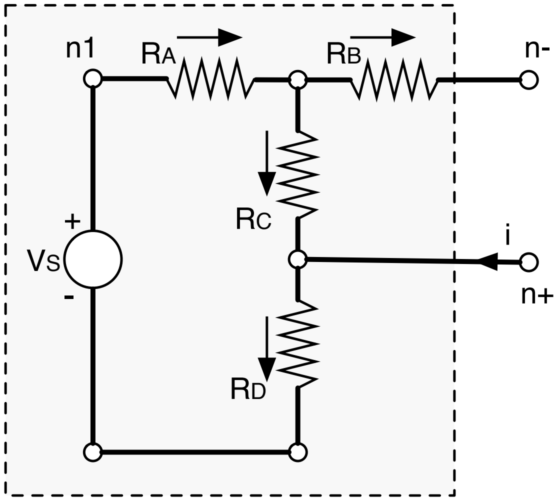
and therefore that

$$\begin{aligned} R_{th} &= -\frac{V_{th}}{i_{sc}} \\ &= V_s \frac{R_B}{R_A + R_B} \frac{V_s}{R_A} \\ &= \frac{R_A R_B}{R_A + R_B} \end{aligned}$$

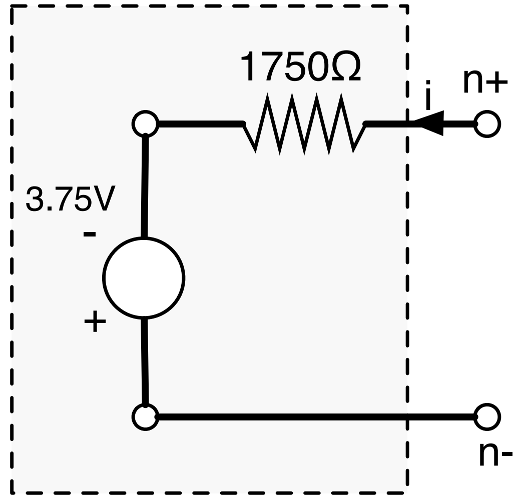
What can we do with this information? We could use it during circuit analysis to simplify parts of a circuit model, individually, making it easier to solve the whole system. We could also use it in design, to construct a simpler implementation of a more complex network design. One important point is that the Thévenin equivalent circuit is not exactly the same as the original one. It will exert the same constraints on the voltages and currents of a circuit that it is connected to, but will, for example, have different heat dissipation properties.

Example

Here's another example, in figure 2(a). It's a bit more hassle than the previous one, but you can write down the equations to describe the constituents and KCL constraints, as before. If we



(a) Same circuit, but with different ports



(b) Its Thévenin equivalent circuit

Figure 3: Thévenin equivalence examples

let $R_A = 2K\Omega$, $R_B = R_C = R_D = 1K\Omega$, and $V_S = 15V$, then we can solve for $V_{th} = 7.5V$ and $R_{th} = 2K\Omega$. So, it is indistinguishable by current and voltage from the circuit shown in figure 2(b).

In figure 3(a) we show the same circuit, but with the connections that run outside the box made to different nodes in the circuit. Note also that the top lead is marked n_- and the bottom one n_+. If we solve, using the same values for the resistors and voltage source as before, we find that $V_{th} = -3.75V$ and $R_{th} = 1750\Omega$. We show the Thévenin equivalent circuit in figure 3(b). We've changed the polarity of the voltage source and made it 3.75V (instead of having the + terminal at the top and a voltage of -3.75), but that's just a matter of drawing.

These results are quite different: so, the moral is, it matters which wires you connect up to what!