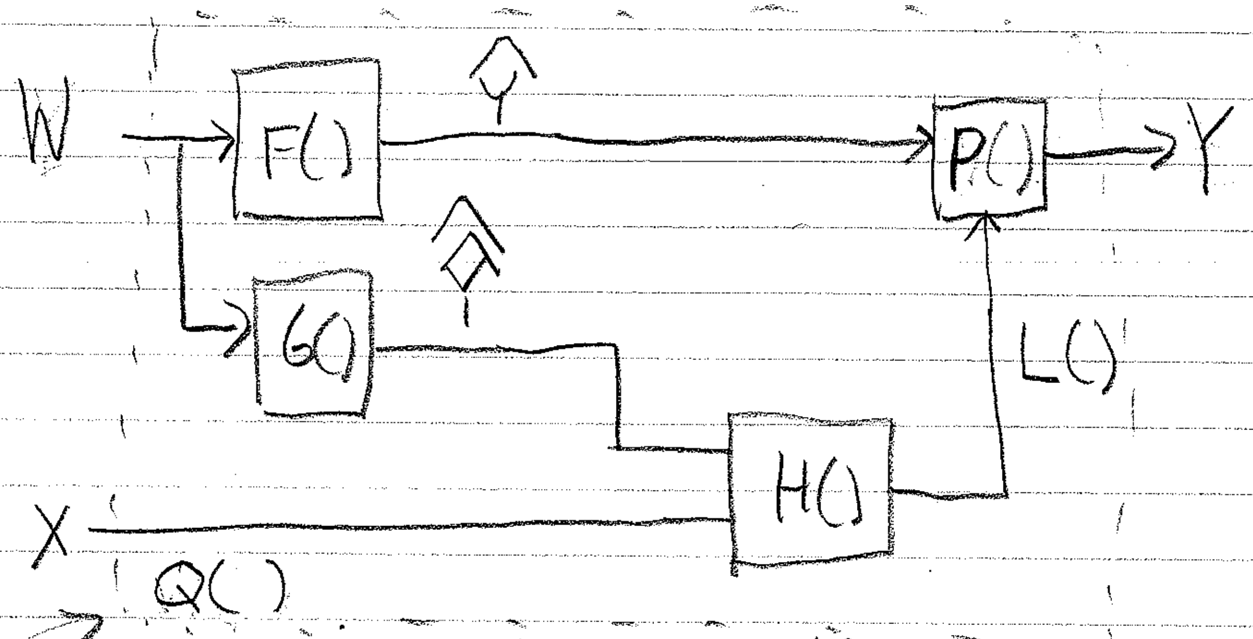
# 6.01 10/16/07



on slides
too much
Front end

<u>In Python:</u>
Create systems with complex behavior by composing functions which consume inputs to generate outputs




Procedures, Class member functions, assignment, *, +, -, %

X * Y → Z
Inputs
output

All take inputs & produce outputs

<u>Hide detail by Abstraction Hierarchy</u> (API)

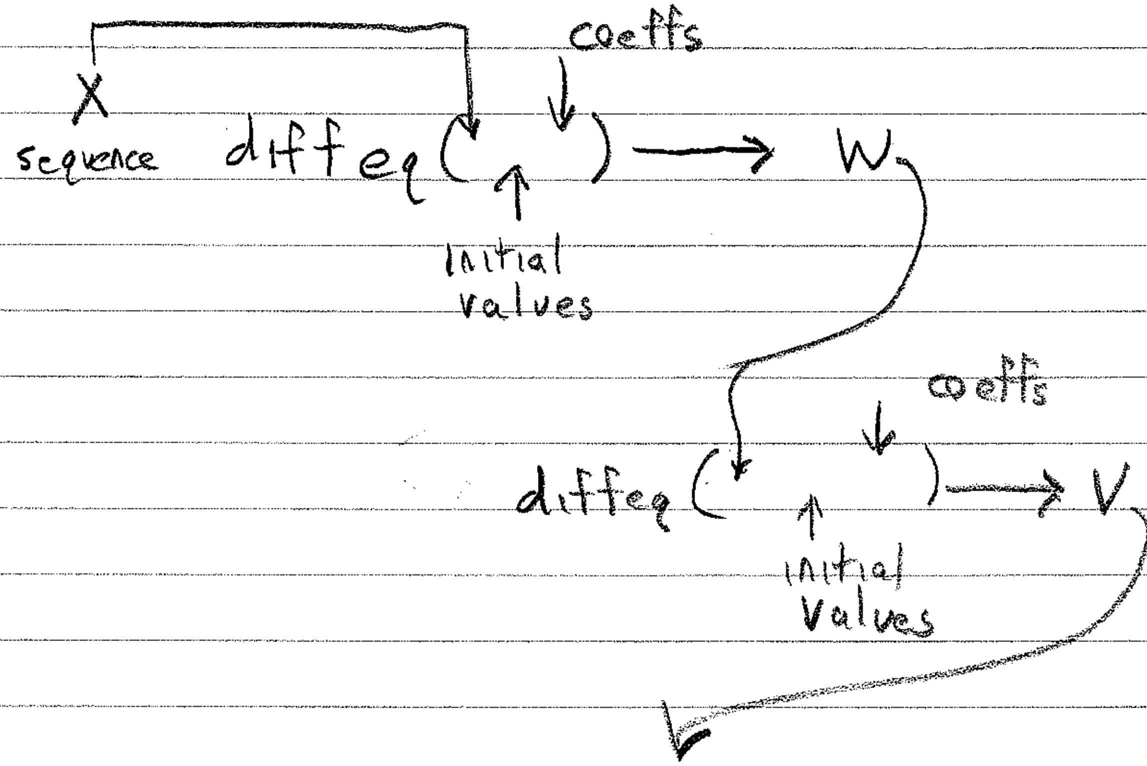
Q(W, X) → Y

## Difference Equation Systems Are also Input-output Descriptions

$$\sum a_k \, w[n+k] = \sum b_\ell \, \overbrace{x[n+\ell]}^{\text{input}}$$

$$\sum c_k \, v[n+k] = \sum d_\ell \, w[n+\ell]$$

$$\sum f_k \, \underbrace{y[n+k]}_{\substack{\text{final} \\ \text{output}}} = \sum g_\ell \, v[n+\ell]$$




All take inputs and produce outputs $\left\{ +, -, \right.$ difference equations, scaling

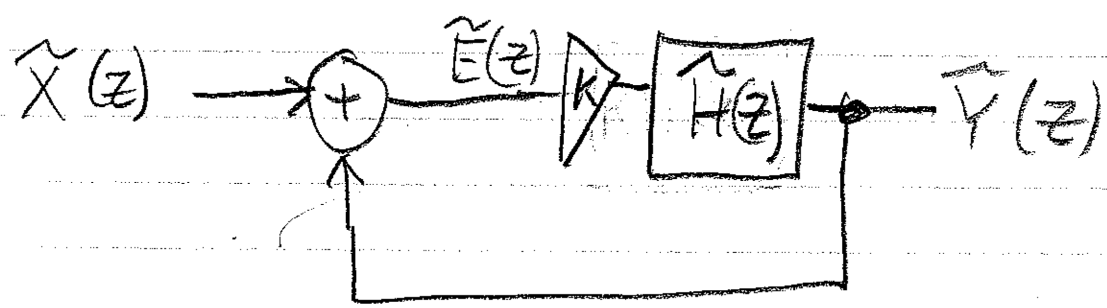
## Introduced an Analysis Tool
### z - transform

$$\tilde{X}(z) \rightarrow \boxed{\tilde{H}_1(z)} \rightarrow \tilde{W}(z) \rightarrow \boxed{\tilde{H}_2(z)} \xrightarrow{\tilde{V}(z)} \boxed{\tilde{H}_3(z)} \rightarrow \tilde{Y}(z)$$

$$\frac{\sum b_\ell z^\ell}{\sum a_k z^k} \qquad \frac{\sum d_\ell z^\ell}{\sum c_k z^k} \qquad \frac{\sum g_\ell z^\ell}{\sum f_k z^k}$$

$$\tilde{X}(z) \longrightarrow \overbrace{\boxed{\tilde{H}_1(z)\,\tilde{H}_2(z)\,\tilde{H}_3(z)}}^{\tilde{H}(z)} \longrightarrow \tilde{Y}(z)$$

$$\frac{\left(\sum b_\ell z^\ell\right)\left(\sum d_\ell z^\ell\right)\left(\sum g_\ell z^\ell\right)}{\left(\sum a_k z^k\right)\left(\sum c_k z^k\right)\left(\sum f_k z^k\right)}$$




<u>Python</u> $X \rightarrow Y$ simple by <u>design</u>

<u>Diff Eq</u> $X \rightarrow Y$ complicated model → need tools

<u>Both Input / Output Views!</u>

Input / Output is a limiting perspective

Unidirectional in Nature

$$X \xrightarrow[\text{generates}]{} Y \xrightarrow[\text{generates}]{} Z \xrightarrow[\text{generates}]{} W$$

Ex

```
def f(x):
    return x**2
```

$y = f(x)$      Set X then get Y

Ex $\quad y(n+1) - y(n) = x(n+1) + x(n)$

$$\tilde{Y}(z) = \left(\frac{z+1}{z-1}\right) \tilde{X}(z)$$

Set $x(n)$ and $y(0)$
then get $y(n)$

Spose Feedback

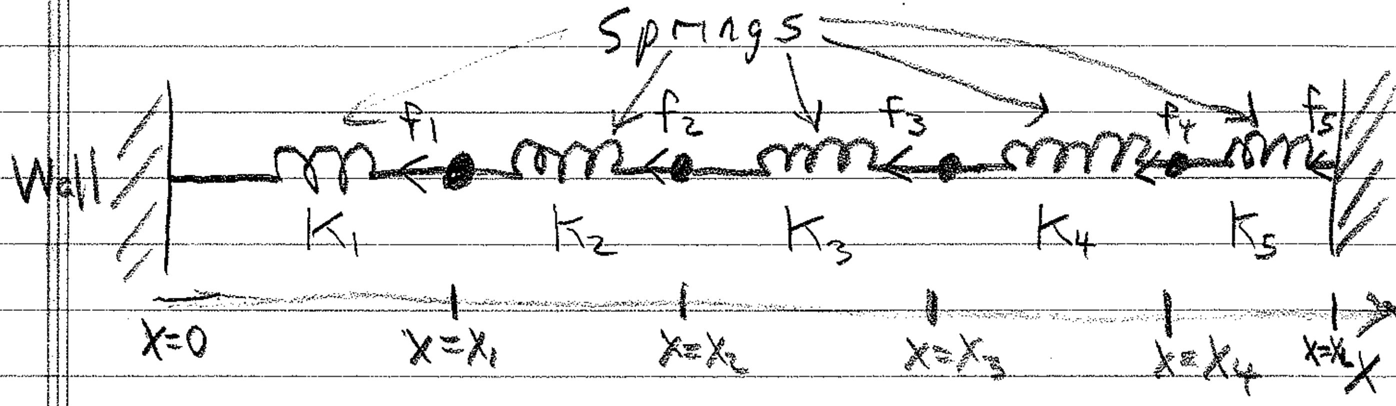
$$e(n) = K(y_d(n) - y(n))$$

$$\tilde{Y}(z) = \frac{z+1}{z-1} \tilde{e}(z) = \frac{z+1}{z-1} K\left(\tilde{Y}_d(z) - \tilde{Y}(z)\right)$$

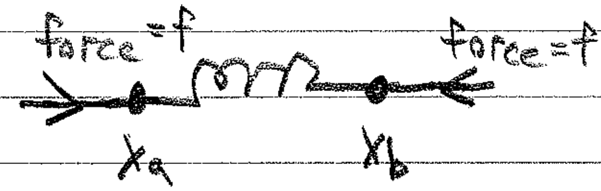
$$\tilde{Y}(z) = \text{————} \tilde{Y}_d(z)$$

Still Unidirectional $\longrightarrow$ Set $y_d(n)$
$\rightarrow$ get $y(n)$

# Consider a set of Springs




Tight Ideal Spring


$x_a$ $x_b$

$x_a - x_b \simeq 0$

$$f = K_{big}(X_b - X_a)$$

Loose Ideal Spring $\quad f = K_{small}(X_b - X_a)$

Input / Output Approach

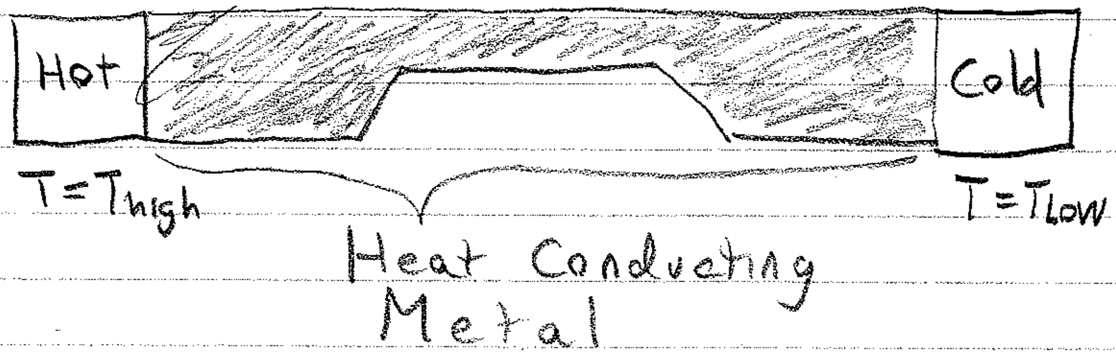
Set $X_1, X_2, X_3, X_4$

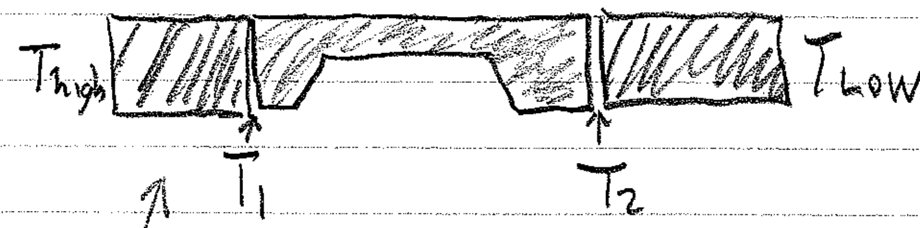
Get $f_1, f_2, f_3, f_4, f_5$

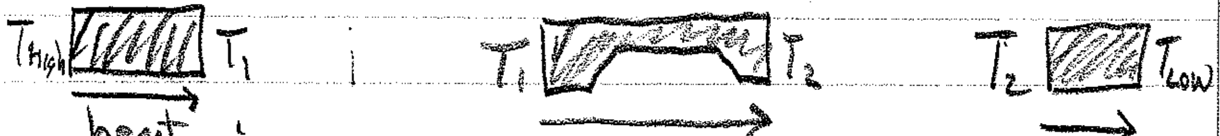
But That's not the question!

Question is: what are the $X$'s so that the force balances?

# Heat Flow




heat flow $= i_{h_1}$ $\qquad i_{h_2} \qquad i_{h_3}$

$i_{h_1} = K_1(T_{high} - T_1)$ $\qquad = K_2(T_1 - T_2)$ $\qquad = K_3(T_2 - T_{LOW})$

Thermal Conductivities

## Input / Output Perspective

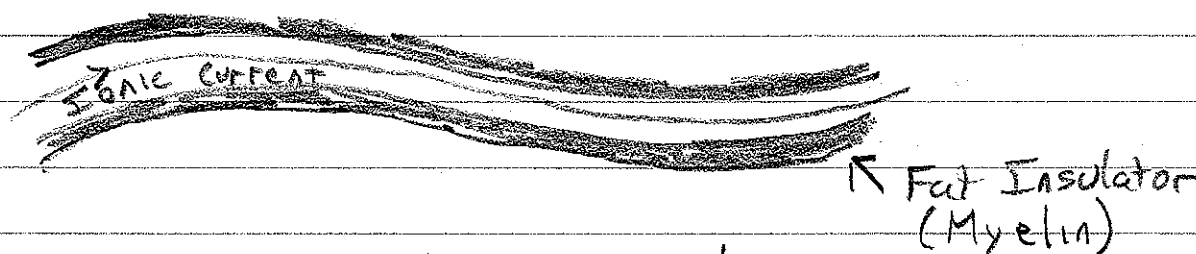
Set $T_1, T_2$

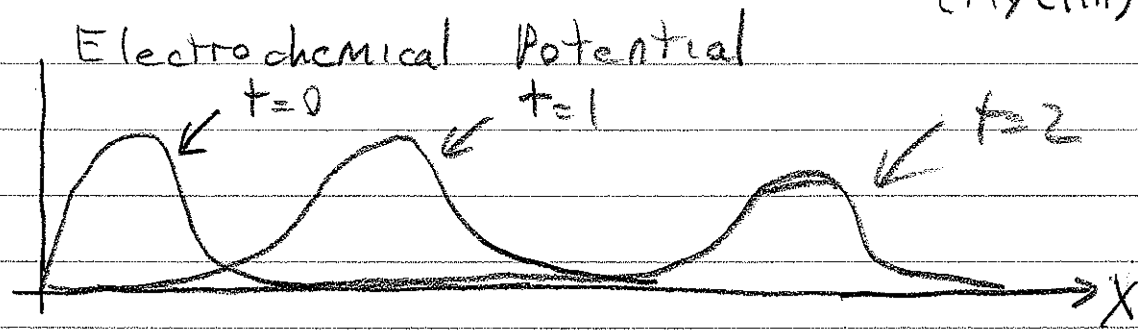
Get $i_{h_1}, i_{h_2}, i_{h_3}$

## Not The right Question!

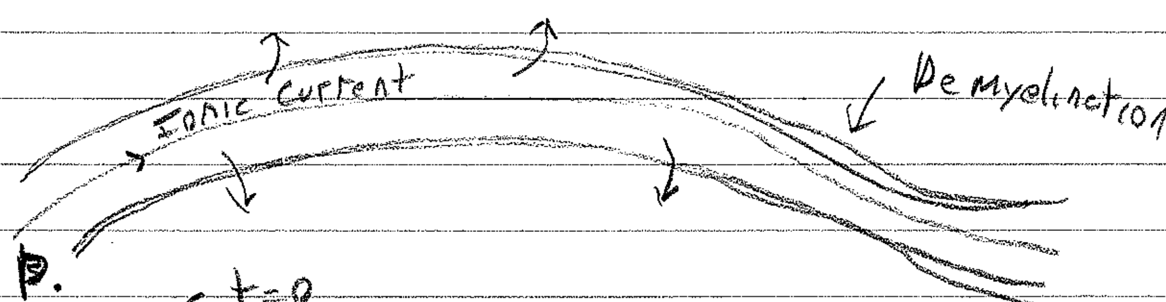
Right Question { What are $T_1$ & $T_2$ so heat flows are Matched?

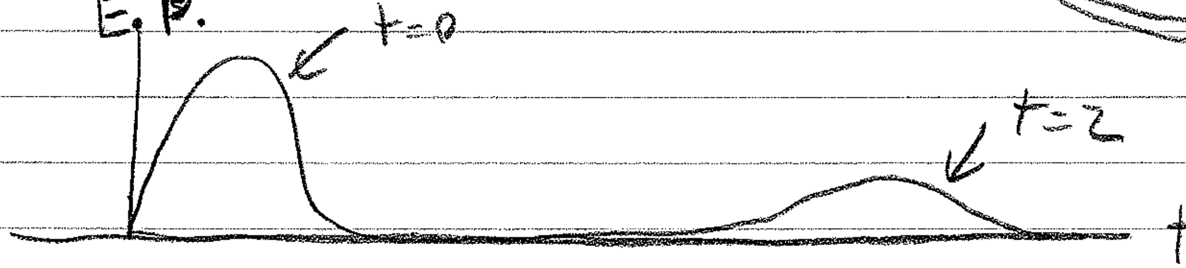
# Signal Conduction in a Nerve




## For Multiple Sclerosis




## Input / Output Perspective

Set E.P.(x)

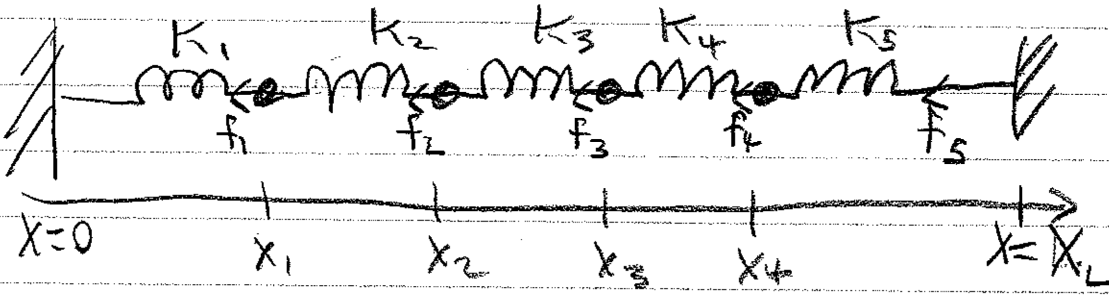
Get Ionic Current(x)

## Not the right Question!

# Examples of Constrained Systems

Springs

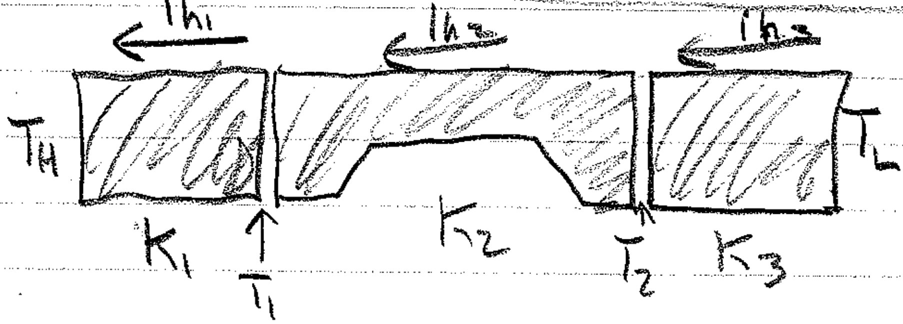
What are $x_1, \dots, x_4$?




Constitutive Relations $\left\{ K_1(x_1 - 0) = f_1 \quad K_2(x_2 - x_1) = f_2 \quad \dots \quad K_5(x_L - x_4) = f_5 \right.$

Conservation Law $\left\{ \underbrace{f_1 - f_2 = 0}_{\text{Force Balance}} \quad f_2 - f_3 = 0 \quad f_3 - f_4 = 0 \quad f_4 - f_5 = 0 \right.$

Heat Flow



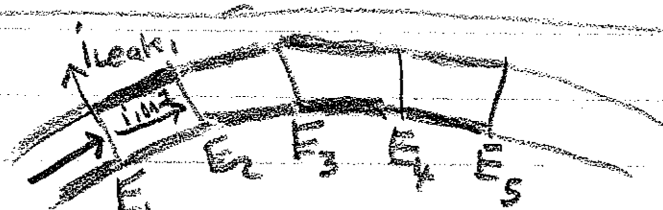

Constit Relate $\left\{ K_1(T_1 - T_H) = i_{h_1} \quad K_2(T_2 - T_1) = i_{h_2} \quad K_3(T_L - T_2) = i_{h_3} \right.$

Conservation Law $\left\{ i_{h_1} - i_{h_2} = 0 \quad i_{h_2} - i_{h_3} = 0 \right.$

Heat Flow Balance

Nerve Conduction

Steady-State Nerve Conduction




$$i_{ion_j} = K_f(E_j - E_{j+1}) \qquad i_{Leak_j} = K_L(E_j)$$

$$i_{ion_j} - i_{ion_{j-1}} + i_{Leak_j} = 0$$

## We will study circuits

Variables: Currents, Voltages

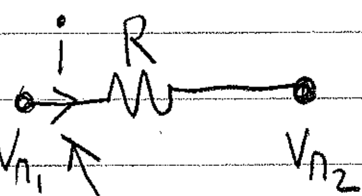
Constraints: $\overbrace{\text{Element voltage-current relations}}^{\text{Constitutive Relations}}$

$\underbrace{\text{Sum of currents at node} = 0}_{\text{Conservation Law}}$

Kirchoff current law (KCL)

## Example Constitutive Relations

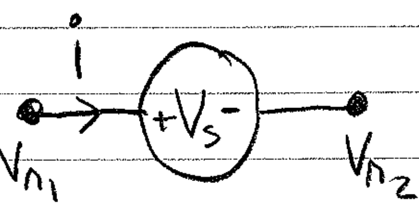
Resistor




Note direction

$$\underbrace{(V_{n_1} - V_{n_2})}_{\text{Note Sign}} - iR = 0$$

Ideal Battery

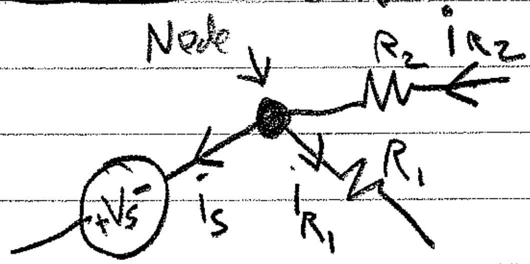



$$(V_{n_1} - V_{n_2}) - V_s = 0$$

## Conservation law

$$\sum_{\text{node}} i\text{'s} = 0$$

Node




$$\underbrace{i_s + i_{R_1}}_{\text{leaving currents positive}} - i_{R_2} = 0$$

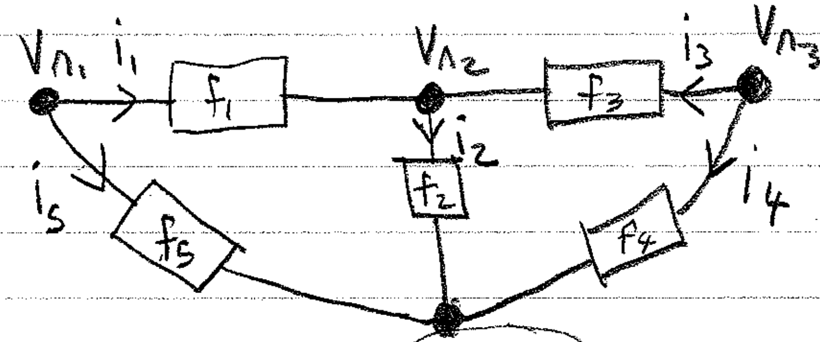
entering current is negative

# Formulating Equations – Nodal Analysis

## Circuit

3 unknown Voltages

5 unknown currents




$V_{n_4} = 0$ ← Reference Voltage

## Constitutive Relations

5 relations

$$f_1(i_1, V_{n_1}, V_{n_2}) = 0$$

$$f_2(i_2, V_{n_2}, V_{n_4}=0) = 0$$

$$f_3(i_3, V_{n_3}, V_{n_2}) = 0$$

$$f_4(i_4, V_{n_3}, V_{n_4}=0) = 0$$

$$f_5(i_5, V_{n_1}, V_{n_4}=0) = 0$$

## Conservation Law

3 relations

$$i_1 + i_5 = 0$$

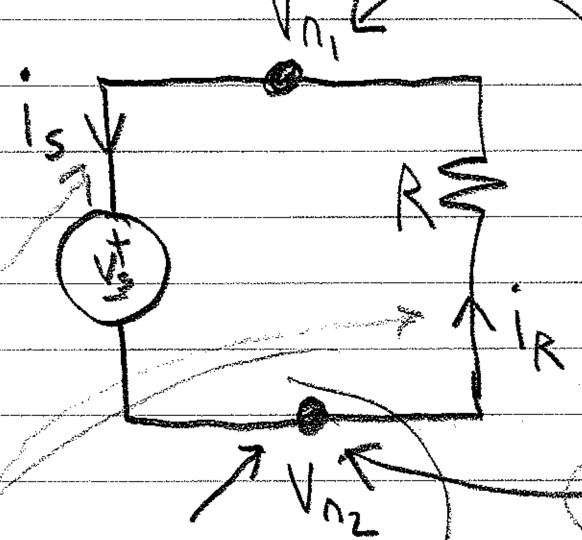
$$-i_3 + i_2 - i_1 = 0$$

$$i_3 + i_4 = 0$$

## NA Algorithm

0) Pick a reference Node
1) Label Element Current Directions
2) Write Constitutive relations for each element
3) Write Conservation Law for each node (except ref)

# Simple Circuit Example




$V_{n_1} - V_{n_2} - V_s = 0$

$R\, i_R - (V_{n_2} - V_{n_1}) = 0$

$i_s - i_R = 0$

$i_R - i_s = 0$

Set $V_{n_2} = 0$

Reference Node

Eliminate extra KCL Equation

**Solve**

$$(V_{n_1} - 0) = V_s$$

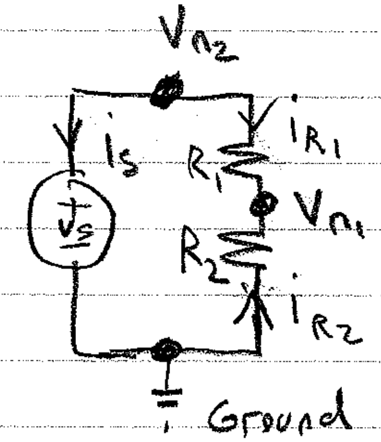
$$R\, i_R - (0 - V_{n_1}) = 0 = R\, i_R + V_s$$

$$i_R = \frac{-V_s}{R}$$

$$i_s = \frac{-V_s}{R}$$

Reverse Directions?

# Voltage Divider Example




**Constit**

$$(V_{n_2} - 0) - V_s = 0$$
$$(V_{n_2} - V_{n_1}) - R_1 i_{R_1} = 0$$
$$(0 - V_{n_1}) - R_2 i_{R_2} = 0$$

**KCL**

$$-i_{R_1} - i_{R_2} = 0$$
$$i_{R_1} + i_s = 0$$

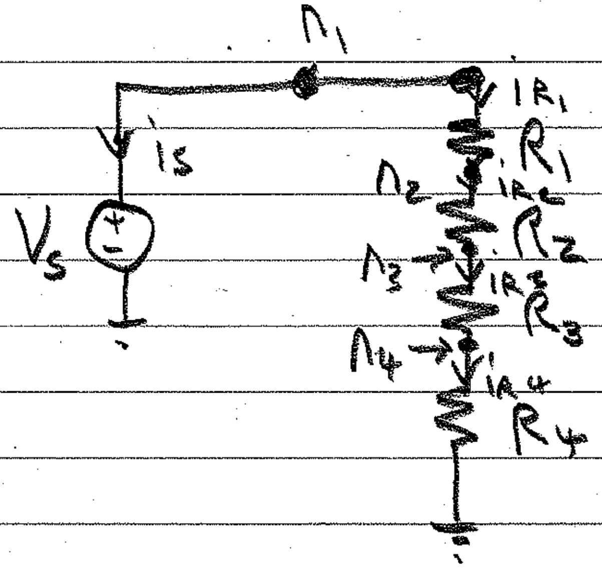
**Solving**

$$\frac{V_s - V_{n_1}}{R_1} \overset{+i_{R_2}}{-i_{R_1}} = 0 \Rightarrow \frac{V_s - V_{n_1}}{R_1} - \frac{V_{n_1}}{R_2} = 0$$

$$(R_1 R_2)\left(+\frac{1}{R_1} + \frac{1}{R_2}\right) V_{n_1} = \left(\frac{V_s}{R_1}\right) R_1 R_2$$

$$\underbrace{V_{n_1} = V_s \frac{R_2}{R_1 + R_2}}_{\text{Voltage Divider Formula}}$$

# Resistors In Series




Crnts Appear Twice

$$i_s + i_{R_1} = 0$$

$$-i_{R_1} + i_{R_2} = 0$$

$$-i_{R_2} + i_{R_3} = 0$$

$$-i_{R_3} + i_{R_4} = 0$$

**At Gnd**

$$-i_s - i_{R_4} = 0$$

$$V_{n_1} - V_{n_2} - i_{R_1} R_1 = 0$$

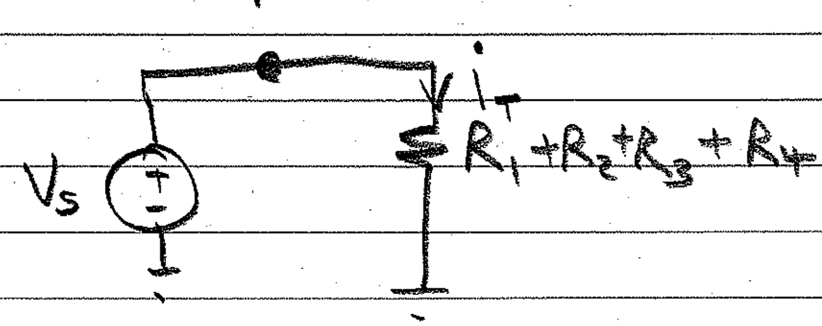
$$V_{n_2} - V_{n_3} - i_{R_2} R_2 = 0$$

$$V_{n_3} - V_{n_4} - i_3 R_3 = 0$$

$$V_{n_4} - 0 - i_4 R_4 = 0$$

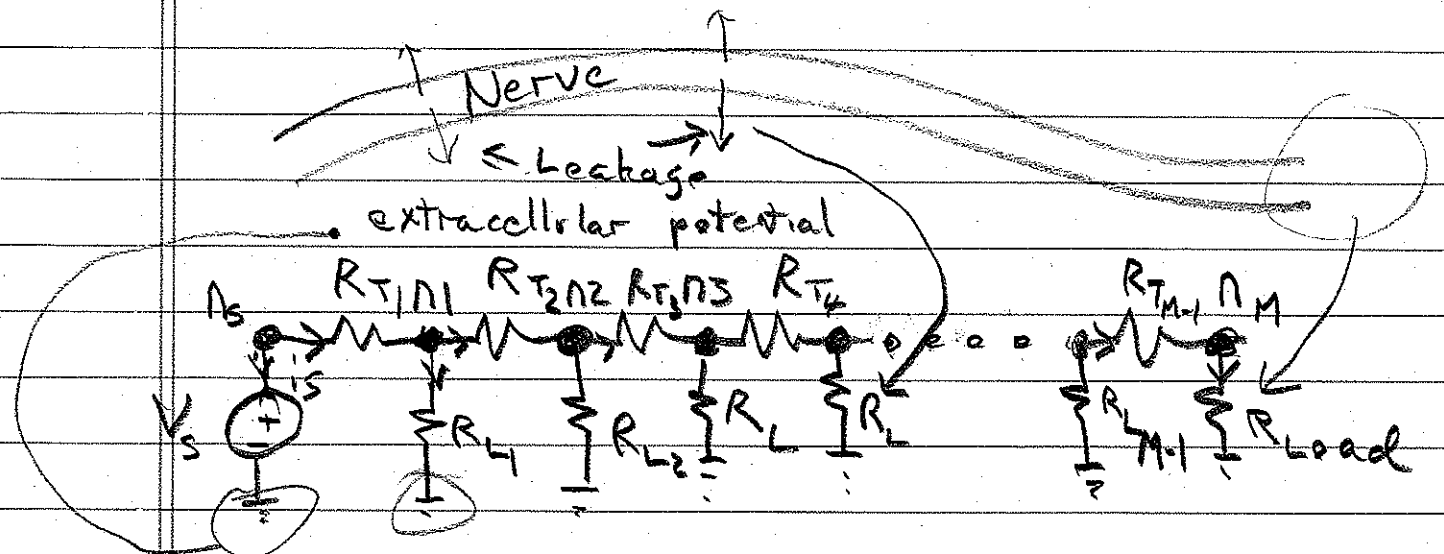
$$i_s = -i_{R_1} = -i_{R_2} = -i_{R_3} = -i_{R_4} = -i_T$$

$$\Rightarrow \quad V_{n_1} - 0 - i_T(R_1 + R_2 + R_3 + R_4) = 0$$




Resistors Add in Series

# Resistor Line




two terminal currents appear in two KCL eqns

$$n_s: \quad i_s + i_{R_{T_1}} = 0$$

$$n_1 \quad -i_{R_{T_1}} + i_{R_{T_2}} + i_{R_{L_1}} = 0$$

$$n_2 \quad -i_{R_{T_2}} + i_{R_{L_2}} + i_{R_{T_3}} = 0$$

$$\vdots$$

$$n_M \quad -i_{R_{T_{M-1}}} + i_{R_{Load}} = 0$$

$i_{R_T}$'s appear twice. $i_{R_L}$'s & $i_s$ appear once. Why?

$$V_{n_s} - 0 - V_s = 0$$

$$V_{n_s} - V_{n_1} - i_{R_{T_1}} R_{T_1} = 0$$

$$V_{n_2} - V_{n_1} - i_{R_{T_2}} R_{T_2} = 0$$

$$\vdots$$

# Resistor Line (Cont'd)

## KCL for Reference Node

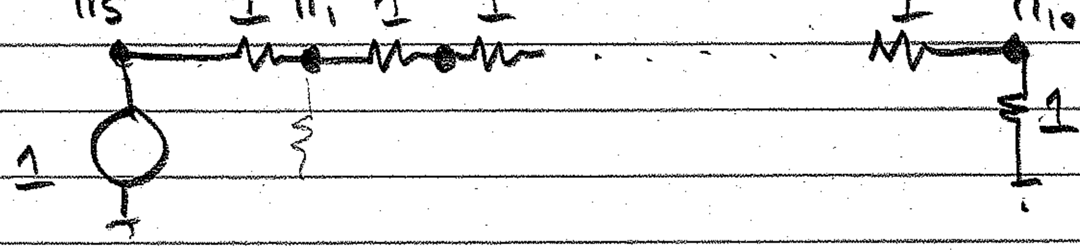
Gnd $\quad -i_s - i_{R_{L1}} - i_{R_2} - \dots - i_{R_{LM-1}} - i_{R_{Load}} = 0$

## Some Observations

Spose $1 = R_{T_1} = R_{T_2} = \dots R_{T_{M-1}} = R_{Load}$

$1000 = R_{L_1} = \dots = R_{L_{M-1}}$

If $V_s = 1, M = 10$, $\simeq$ what is $V_{n_M}$




If $R_{L_s} = 1$, what is $V_{n_M}$?