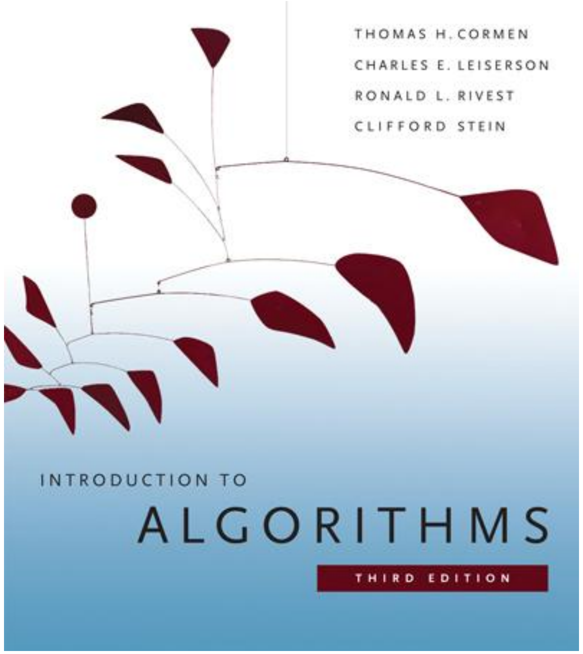
6.006- *Introduction to Algorithms*



Lecture 2

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Menu

- Problem: peak finding
  - 1 dimension
  - 2 dimensions

- Technique: *Divide and conquer*
- *details about the 1st pset in the end of the lecture*

Peak Finding: 1D

- Consider an array $A[1\ldots n]$:

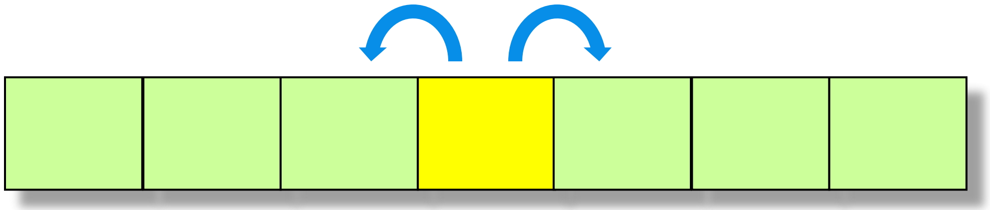
10	13	5	8	3	2	1

- An element $A[i]$ is a *peak* if it is not smaller than its neighbor(s). I.e.,
  - if $i \neq 1, n$: $A[i] \geq A[i-1]$ and $A[i] \geq A[i+1]$
  - If $i=1$: $A[1] \geq A[2]$
  - If $i=n$: $A[n] \geq A[n-1]$
- Problem: find ***any*** peak.

Peak Finding: Ideas ?

- Algorithm I:
  - Scan the array from left to right
  - Compare each A[i] with its neighbors
  - Exit when found a peak
- Complexity:
  - Might need to scan all elements, so $T(n)=\Theta(n)$

Peak Finding: Ideas II ?



- Algorithm II:
- Consider the middle element of the array and compare with neighbors
  - If A[n/2-1]>A[n/2]

    then search for a peak among A[1]… A[n/2-1]
  - Else, if A[n/2]<A[n/2+1]

    then search for a peak among A[n/2]… A[n]
  - Else A[n/2] is a peak!

    (since $A[n/2-1] \leq A[n/2]$ and $A[n/2] \geq A[n/2+1]$)
- Running time ?

Algorithm II: Complexity

Algorithm II: Complexity

Time needed to find peak in array of length n

Recursion

Time for comparing A[n/2] with neighbors

- We have

$$T(n) = T(n/2) + \Theta(1)$$

- Unraveling the recursion,

$$T(n) = \underbrace{\Theta(1) + \Theta(1) + \ldots + \Theta(1)}_{\log_2 n} = \mathbf{\Theta(\log n)}$$

- $\log n$ is much much better than n !

Divide and Conquer

- Very powerful design tool:
  - *Divide* input into multiple disjoint parts
  - *Conquer* each of the parts separately (using recursive call)
- Occasionally, we need to *combine* results from different calls (not used here)

Peak Finding: 2D

- Consider a 2D array A[1…n, 1…m] :

10	8	5
3	2	1
7	13	4
6	8	3

- An element A[i] is a *2D peak* if it is not smaller than its (at most 4) neighbors.
- Problem: find any 2D peak.

2D Peak Finding: Ideas?

Algorithm I: use the 1D algorithm

- Algorithm I:
  - For each column j, find its *global* maximum B[j]
  - Apply 1D peak finder to find a peak (say B[j]) of B[1...m]
- Running time ?

…is $\Theta(n \cdot m)$

- Correctness:
  - B[j] not smaller than B[j-1], B[j+1]
  - For any k, B[k] not smaller than any element from the k-th column of A
  - Therefore, B[j] not smaller than any element from the columns j-1, j and j+1 of A
  - But this includes all neighbors of B[j] in A, so B[j] is a peak in A

12	8	5
11	3	6
10	9	2
8	4	1

12	9	6

Algorithm I': use the 1D algorithm

- Observation: 1D peak finder uses only O(log m) entries of B
- We can modify Algorithm I so that it only computes B[j] *when needed* !
- Total time ?

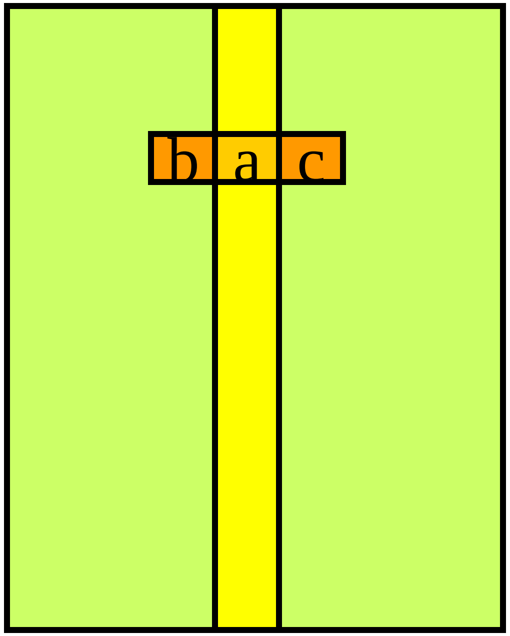
  …only O(n log m) !
  - Need O(log m) entries B[j]
  - Each computed in O(n) time

12	8	5
11	3	6
10	9	2
8	4	1

12	9	6

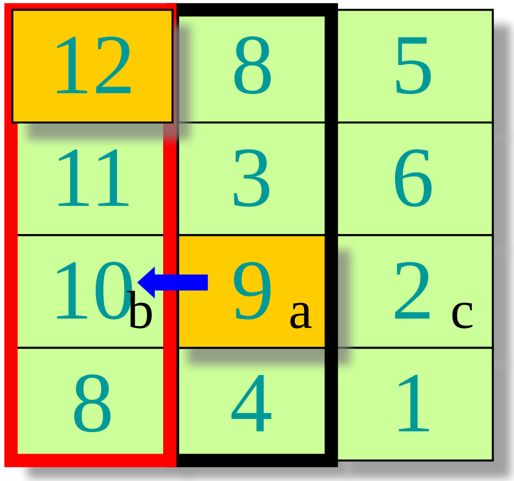
Algorithm II

- Pick middle column ($j=m/2$)
- Find *global* maximum $a=A[i,m/2]$ in that column
  (and quit if $m=1$)
- Compare a to $b=A[i,m/2-1]$ and $c=A[i,m/2+1]$
- If $b>a$
  then recurse on left columns
- Else, if $c>a$
  then recurse on right columns
- Else a is a 2D peak!



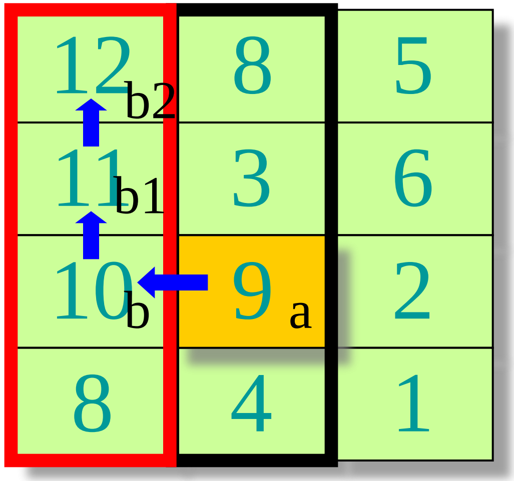
Algorithm II: Example

- Pick middle column (j=m/2)
- Find *global* maximum a=A[i,m/2] in that column
  (and quit if m=1)
- Compare a to b=A[i,m/2-1] and c=A[i,m/2+1]
- If b>a
  then recurse on left columns
- Else, if c>a
  then recurse on right columns
- Else a is a 2D peak!



Algorithm II: Correctness

- Claim: If b>a, then there is a peak among the left columns
- Proof (by contradiction):
  - Assume no peak on the left
  - Then b must have a neighbor b1 with higher value
  - And b1 must have a neighbor b2 with higher value
  - …
  - We have to stay on the left side – why?
  - (because we cannot enter the middle column)
  - But at some point, we would run out the elements of the left columns
  - Hence, we have to find a peak at some point



Algorithm II: Complexity

- We have

$$T(n,m) = \underbrace{T(n,m/2)}_{\text{Recursion}} + \underbrace{\Theta(n)}_{\text{Scanning middle column}}$$

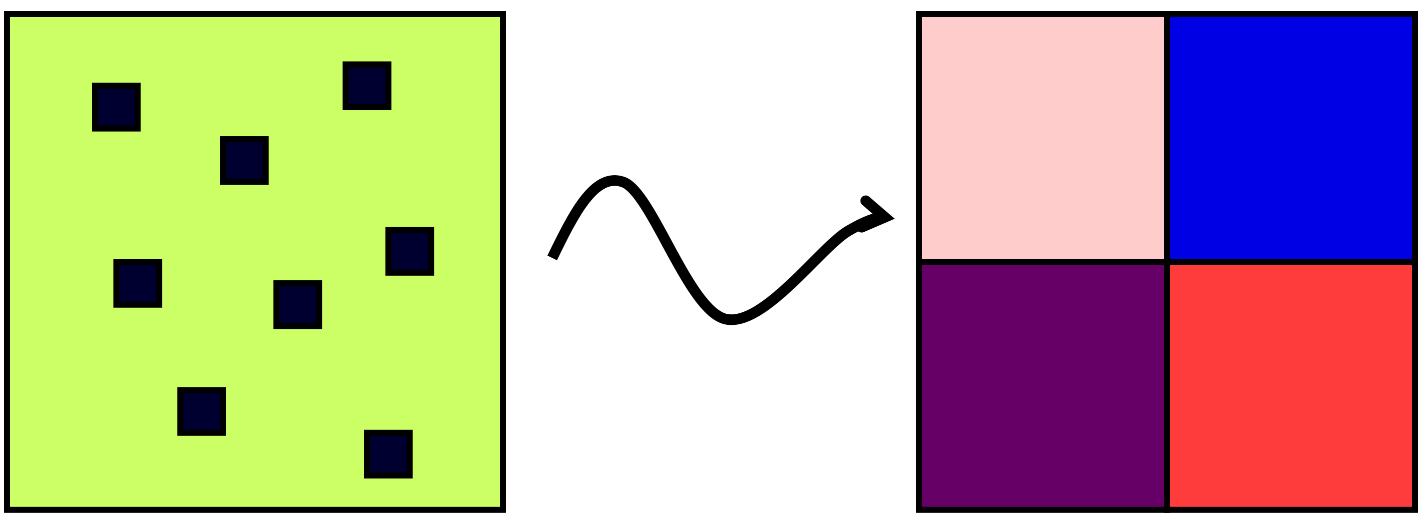
- Hence:
  - $T(n,n) = \underbrace{\Theta(n) + \Theta(n) + \ldots + \Theta(n)}_{\log_2 m} = \mathbf{\Theta(n\log m)}$

Faster than O(n log n) ?

- Idea:

  Reading only O(n + m) elements, reduce an array of $n \times m$ candidates to an array of $n/2 \times m/2$ candidates

- Pictorially:



read only O(n +m) elements

Faster than O(n log n) ?

- Hypothetical algorithm has recursion:

$$T(n,m) = T\left(\frac{n}{2}, \frac{m}{2}\right) + \Theta(n+m)$$

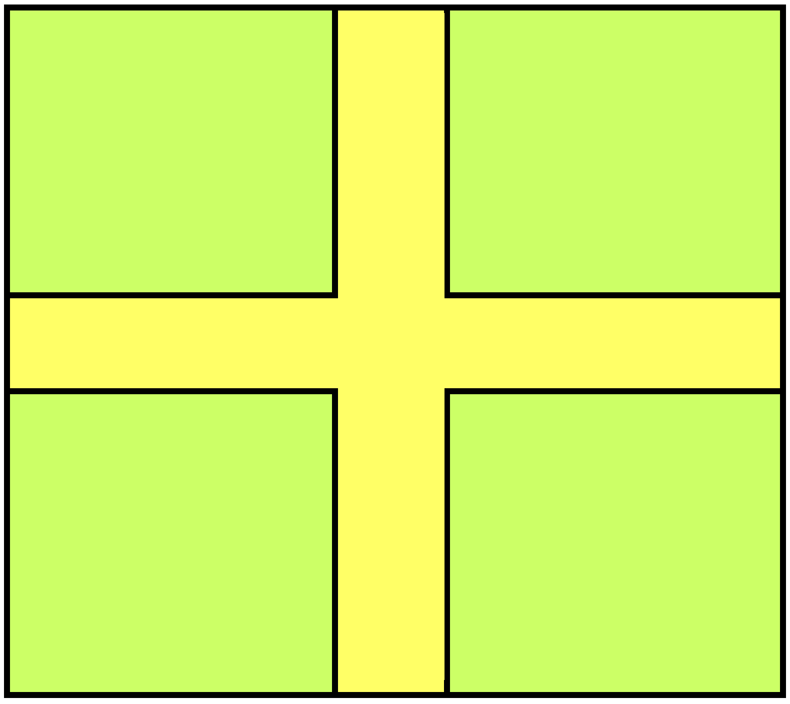
- Hence:

$$\begin{aligned} T(n,m) = \Theta(n+m) + \Theta\left(\frac{n+m}{2}\right) \\ +\Theta\left(\frac{n+m}{4}\right) \\ + \ldots + \Theta(1) \end{aligned}$$

$$= \Theta(n+m)$$ **!**

Towards a linear-time algorithm

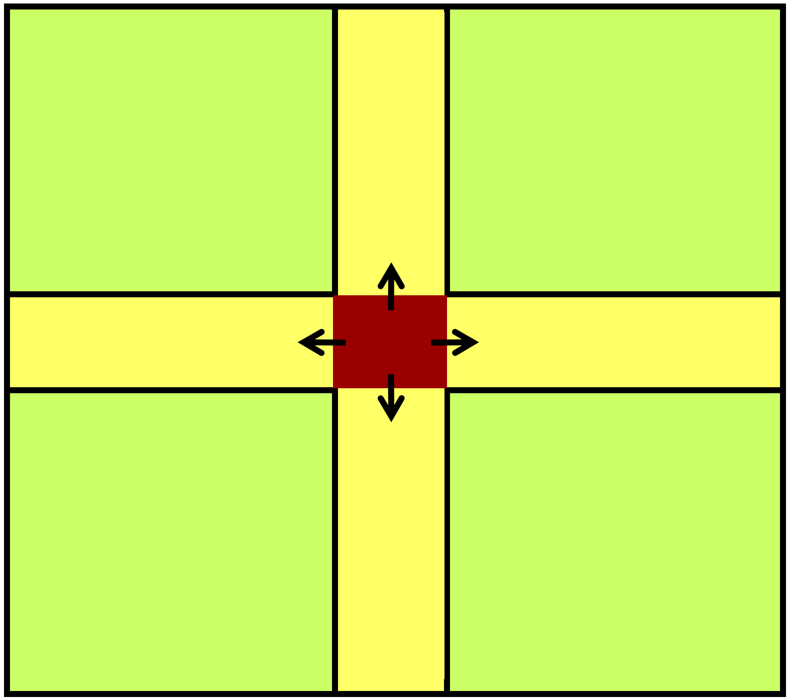
What elements are useful to check?



- suppose we find global max on the cross

Towards a linear-time algorithm

What elements are useful to check?

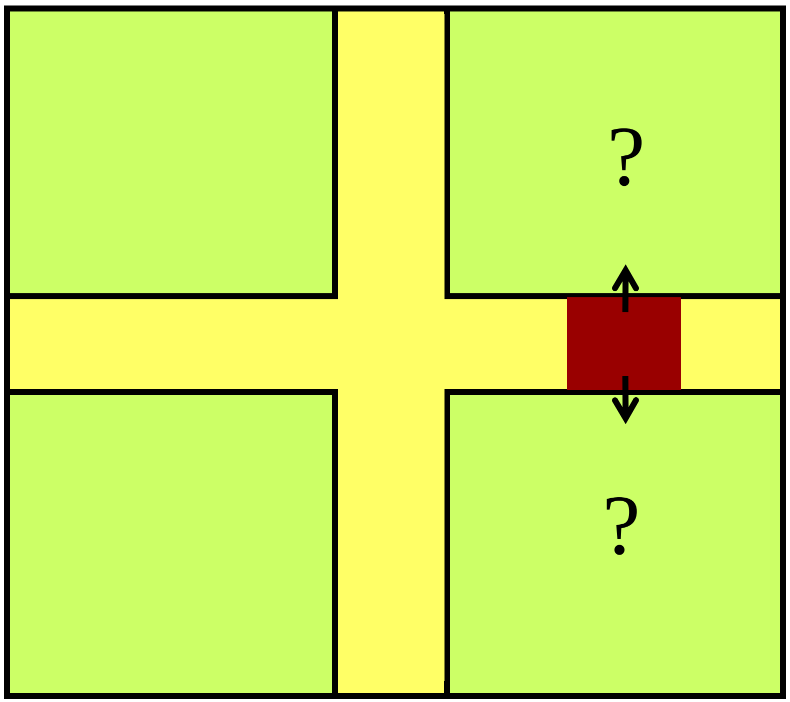


- suppose we find global max on the cross
- if middle element done!

Towards a linear-time algorithm

What elements are useful to check?

proof of claim 2 and fix to this algorithm provided in recitation



- find global max on the cross

- if middle element done!

- o.w. two candidate sub-squares

- determine which one to pick by looking at its neighbors not on the cross (as in Algorithm II)

Claim: The sub-square chosen by the above procedure (if any), always contains a peak of the large square.

BUT: **Claim 2:** Not every peak of the chosen sub-square is necessarily a peak of the large square. Hence, it is hard to recurse…

First Problem Set

- out tonight, by 9pm
  - part A: theory, due at 11.59pm, Sept 21st
  - part B: implementation, due at 11.59pm, Sept 23rd
- deadline policy:
  - 6 days of credit can be used for delayed homework submission
  - at most 2 days can be used for the same deadline (total of 12 deadlines: 6psets x 2parts)
- details on the class website