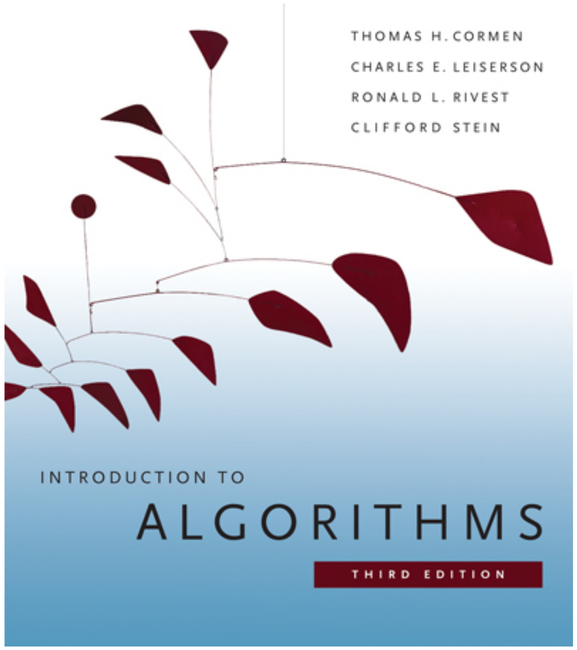
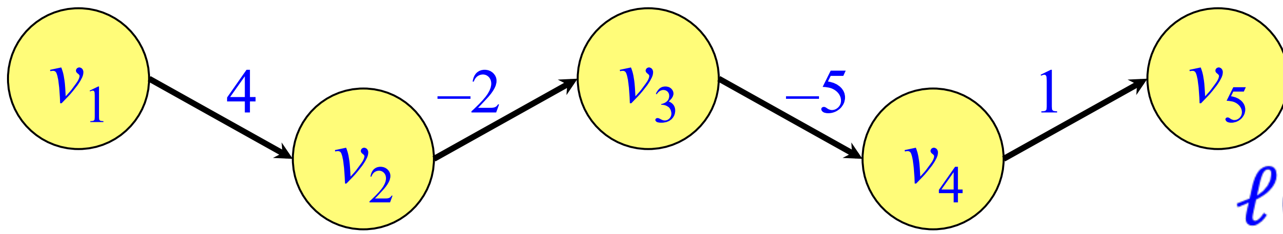
6.006- *Introduction to Algorithms*



Lecture 15

Prof. Silvio Micali

Shortest Paths in a Graph



$\ell(p) = -2$

- $\ell(e)$ length of edge e
- $\ell(p)$ length of path p

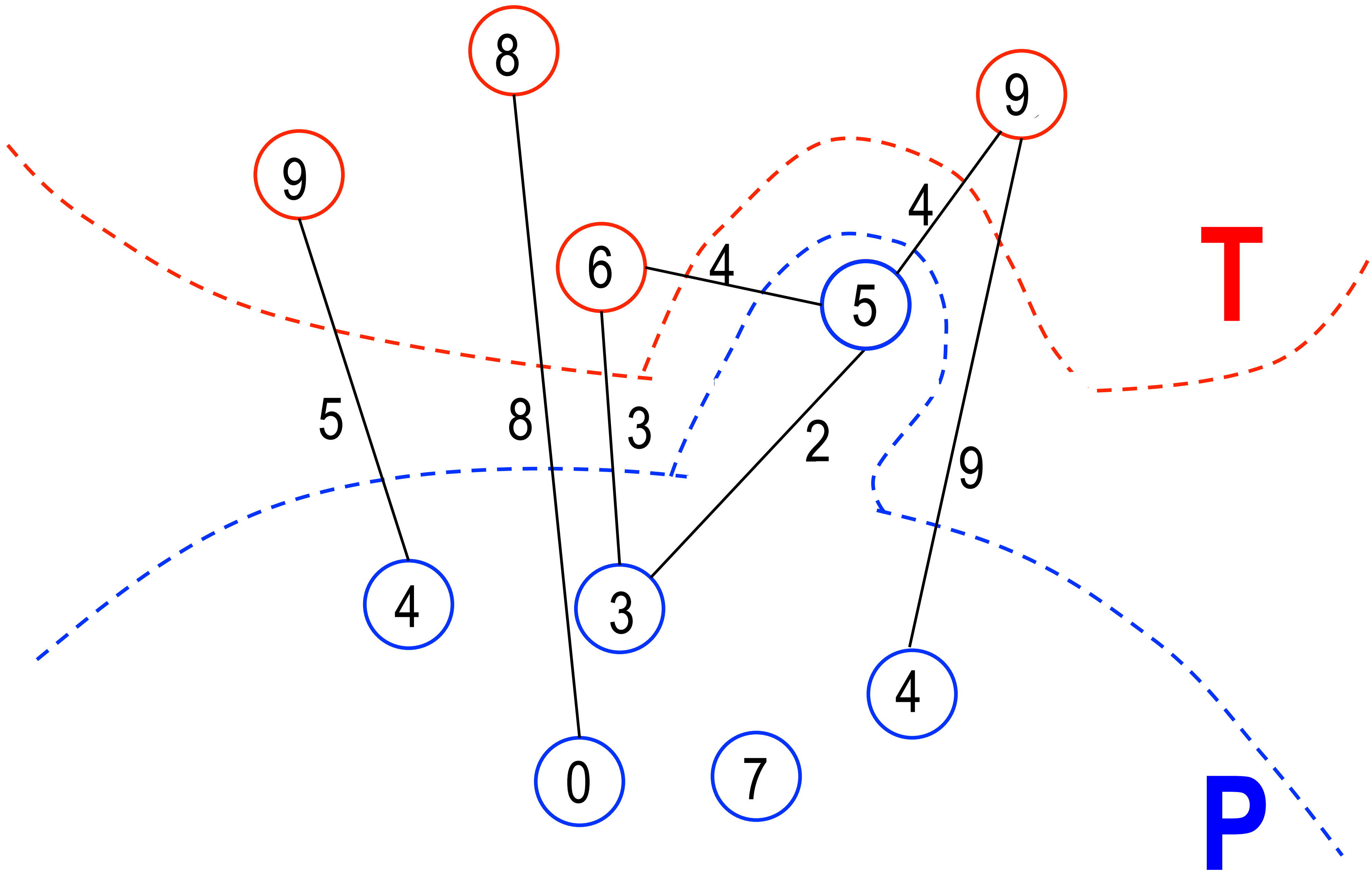
Given a "source" s:

- $\lambda(v)$ $\exists$ path p from s to v with $\ell(p) = \lambda(v)$
  You can get from s to v in $\lambda(v)$ *length*

- $\delta(v)$ distance from s to v
  Length of **a** shortest path from s to v

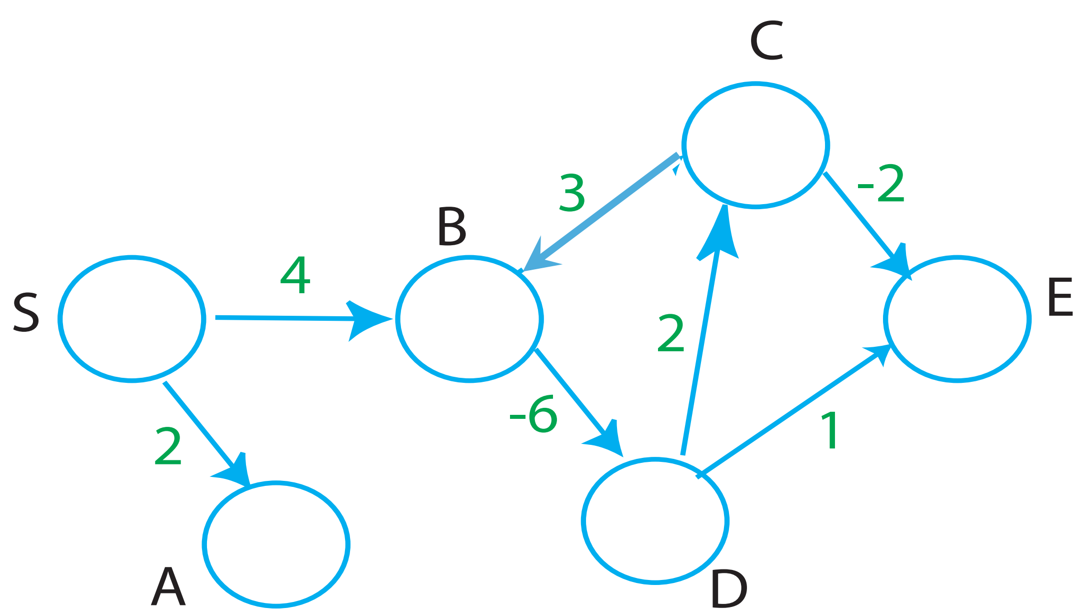
Recall: Dijstra’s Algorithm

Distances from s when $\ell: E \to [0,+\infty]$


8
9
9
6
4
5
4
5
8
3
2
9
4
3
4
0
7
T
P

Today: Shortest Paths (from s) in Digraphs with General Edge Length $\ell : E \to (-\infty, +\infty)$

Makes sense when G has no negative cycles!



Notation

- $\ell(e)$ length of edge e
- $\ell(p)$ length of path p

Given a "source" s:

- $\lambda(v)$ $\exists$ path p from s to v with $\ell(p) = \lambda(v)$

- $\delta(v)$ distance from s to v

- $\pi(v)$ predecessor of v on a best path so far
  (initially, $\pi(s) = s$, and $\pi(v) = NIL \;\; v \neq s$

A generic start

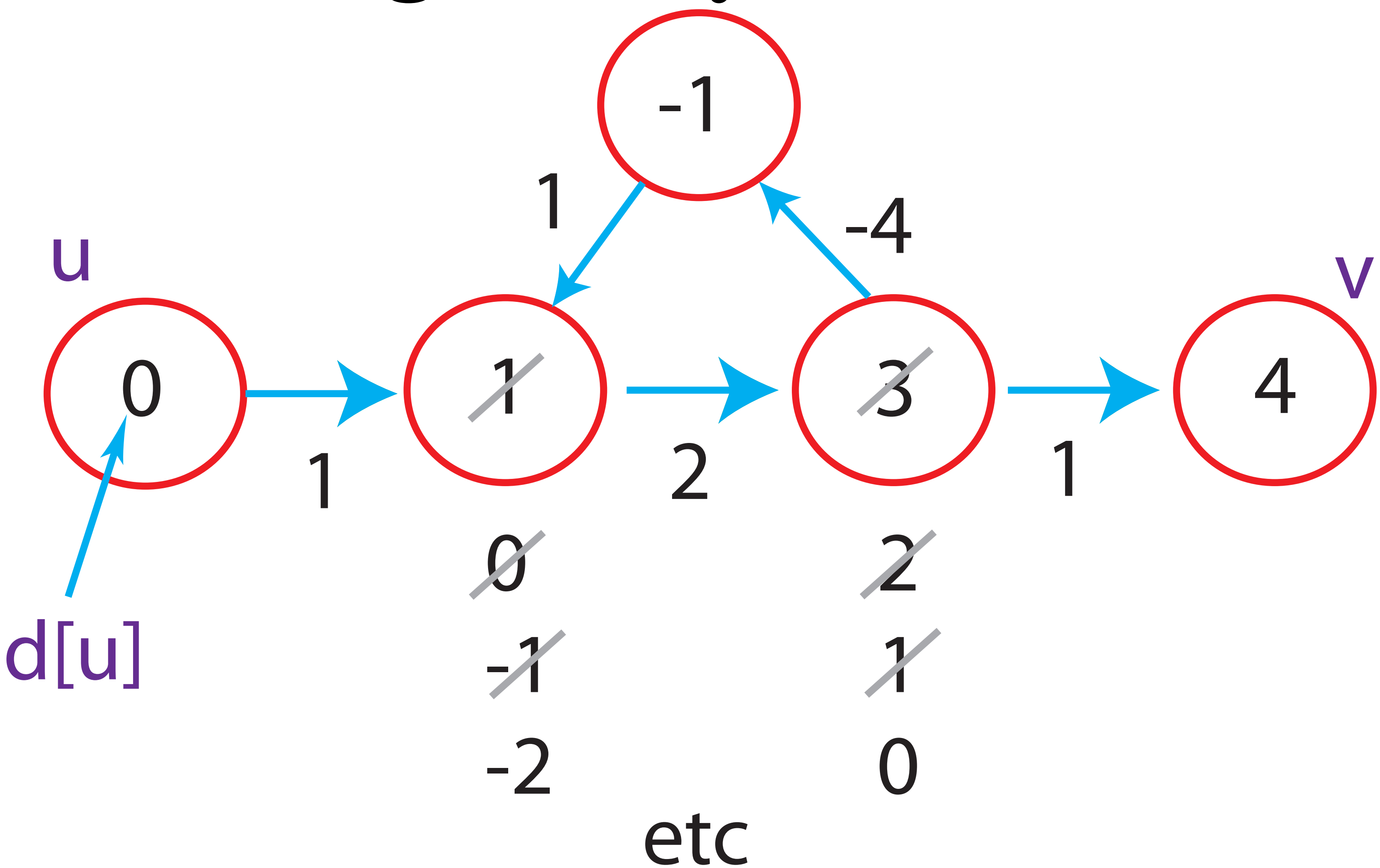
$d[s] \leftarrow 0$
$\pi[s] \leftarrow \mathrm{s}$
for each $v \in V - \{s\}$
  do $d[v] \leftarrow \infty$
    $\pi[v] \leftarrow \mathrm{nil}$

initialization

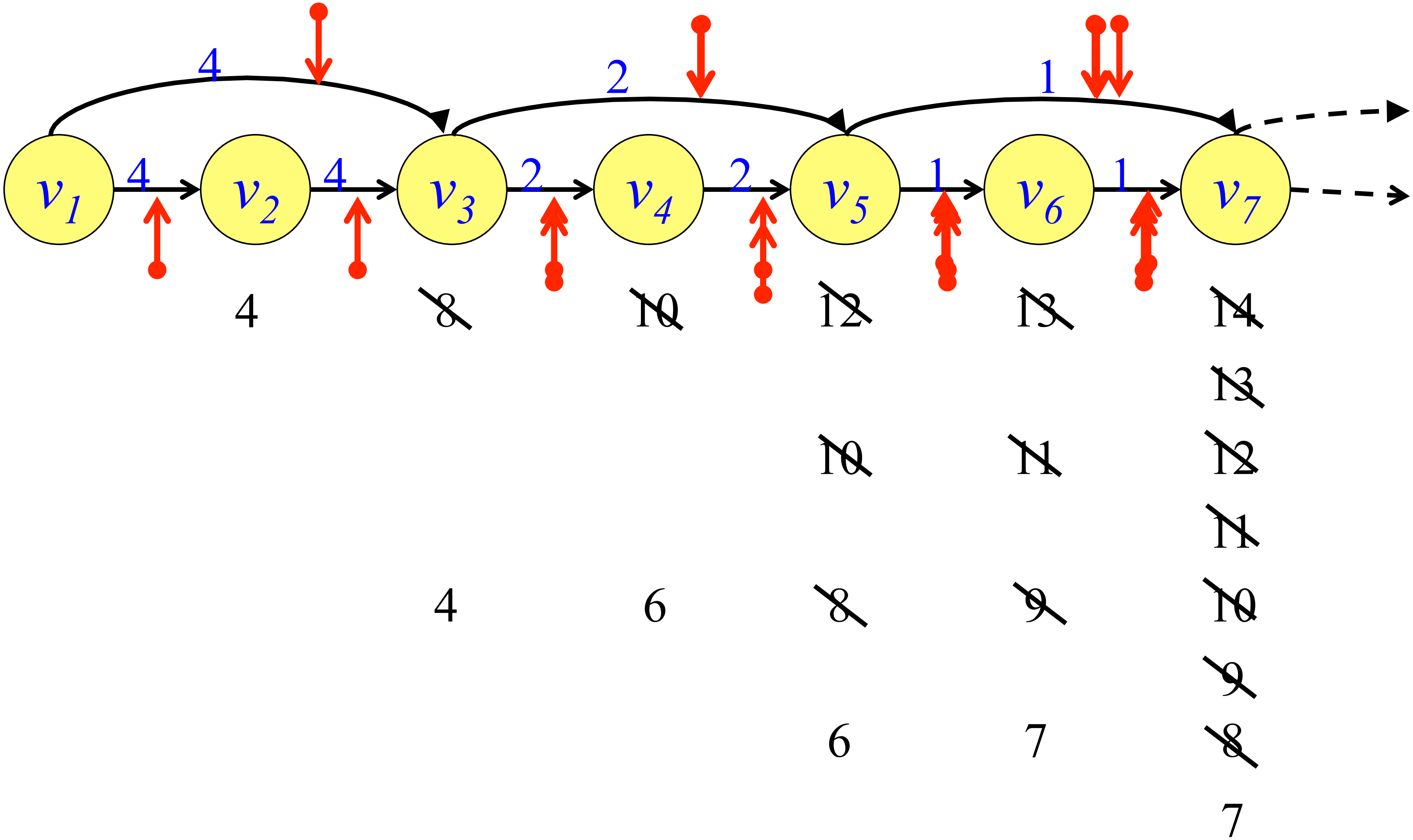
while there is an edge $(u, v) \in E$ s. t.
  $d[v] > d[u] + \ell(u, v)$ **do**
 select arbitrarily one such edge
 set $d[v] \leftarrow d[u] + \ell(u, v)$
  $\pi[v] \leftarrow \mathrm{u}$
endwhile

Relaxation (Improvement) Step

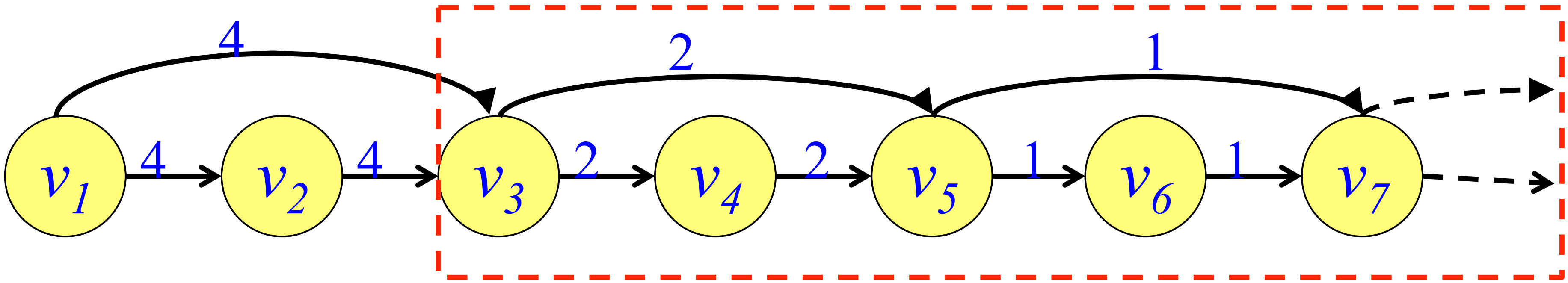
Of course, it will not stop when negative cycles exist



What if no negative cycle ....



What if no negative cycle ....



Analysis = # of relaxations

$T(n)$? n = number of vertices

$T(n) = 3 + 2T(n-2)$ $\Rightarrow T(n) = \Theta(2^{\frac{n}{2}})$

Need to be **careful** how you relax!

HOW? (Bellman Ford)

- Arbitrarily fix an ordering of the edges: $e_1, \dots, e_m$

0. $\lambda(s) \leftarrow 0$ $\qquad\qquad \forall x \neq s$: $\lambda(x) = +\infty$

1. Until no improvement found do:

PASS $\stackrel{\text{def}}{=}$

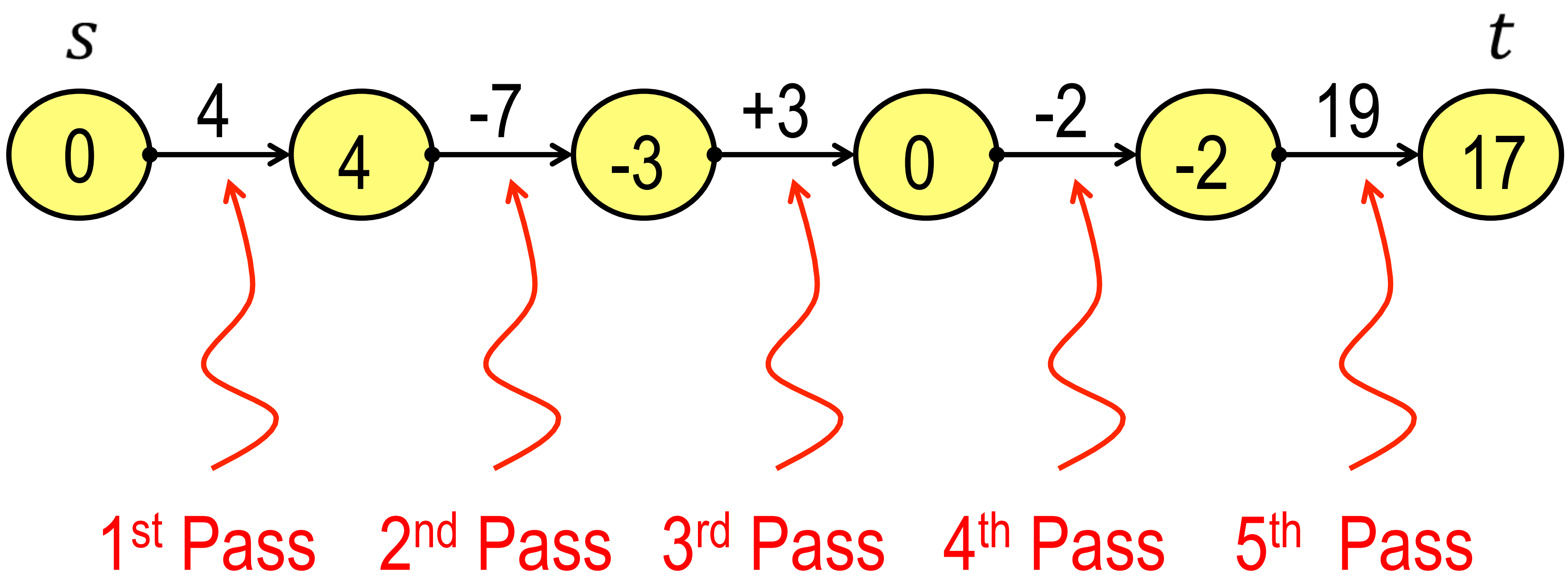
$$For\ i = 1\ to\ m\ do:$$

IF $u \xrightarrow{e_i} v$ is such that $\lambda(v) > \lambda(u) + \ell(e_i)$

Then $\lambda(v) \leftarrow \lambda(u) + \ell(e_i)$

Cost of one PASS = $O(m)$ How many PASSES ?

Let this be shortest path from s to t



Ford's Total Complexity = $O(nm)$

And if G=(V,E) had cycles?

Take Homes

Optimal substructure

Theorem. A subpath of a shortest path is a shortest path.

Proof. By contradiction ...

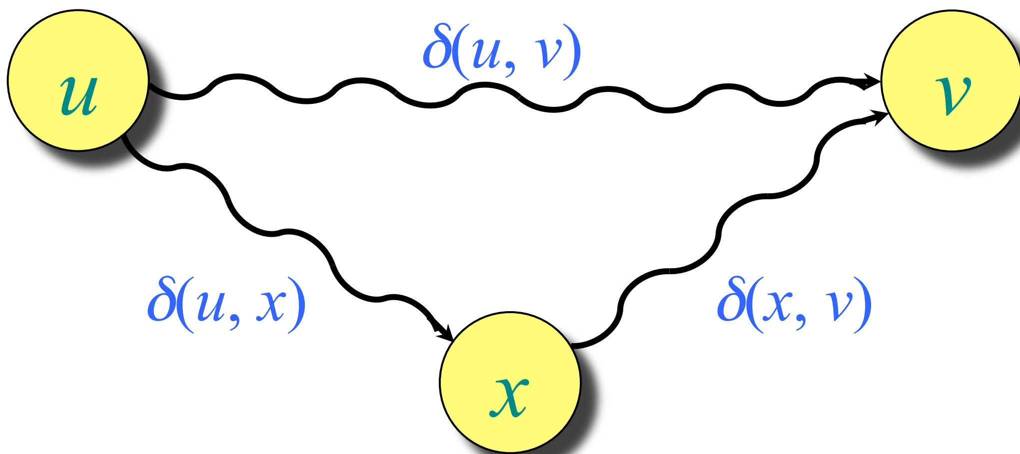
$$\begin{tikzcd} p = v_0 \arrow[r, squiggly, "p_{0j}"] & v_i \arrow[r, "p_{ij}"] \arrow[r, bend right=60, dashed, "p_{ij}'"'] & v_j \arrow[r, squiggly, "p_{jk}"] & v_k \end{tikzcd}$$

Triangle inequality

Theorem. For all $u, v, x \in V$, we have

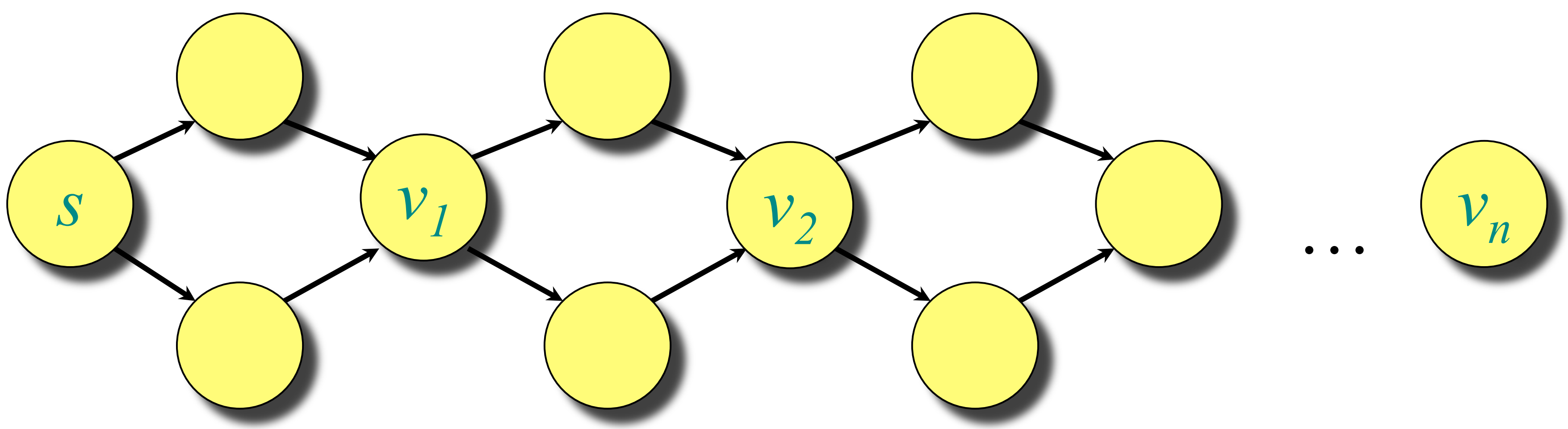
$$\delta(u, v) \le \delta(u, x) + \delta(x, v).$$

Proof.



∎

Combinatorics vs. Combinatorial Optimization



"See you laters alligaters"