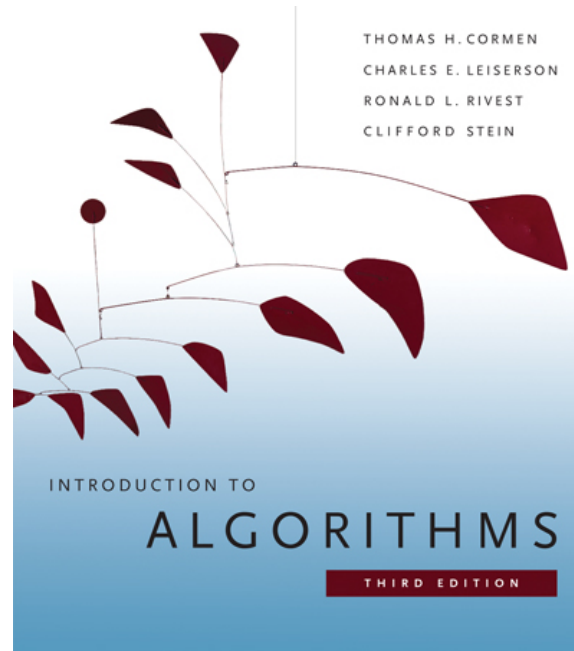


6.006- *Introduction to Algorithms*

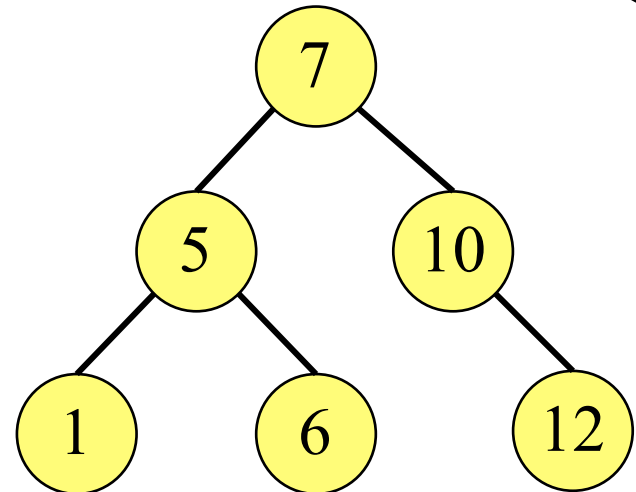
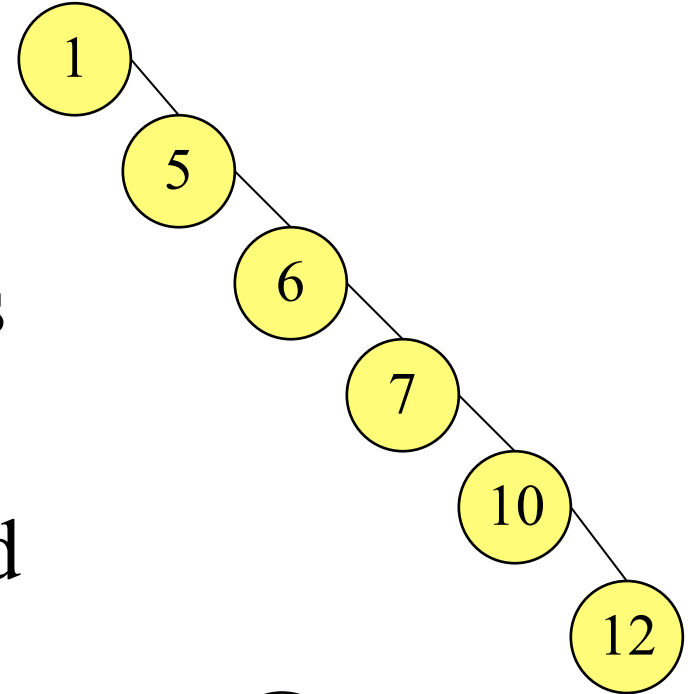


Lecture 4

Prof. Silvio Micali

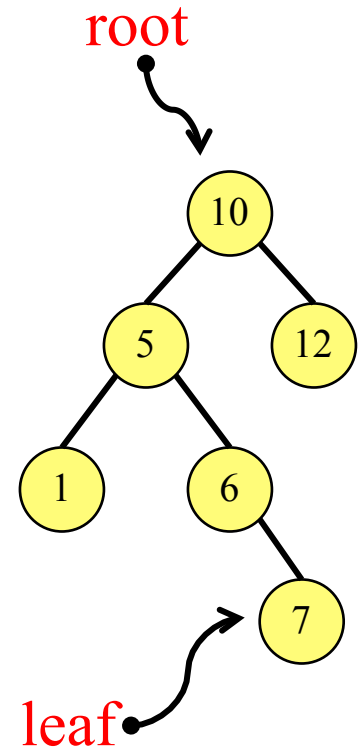
Lecture Overview

- Review: Binary Search Trees
- Importance of being balanced
- Balanced BSTs
 - AVL trees
 - Other balanced trees



Binary Search Trees (BSTs)

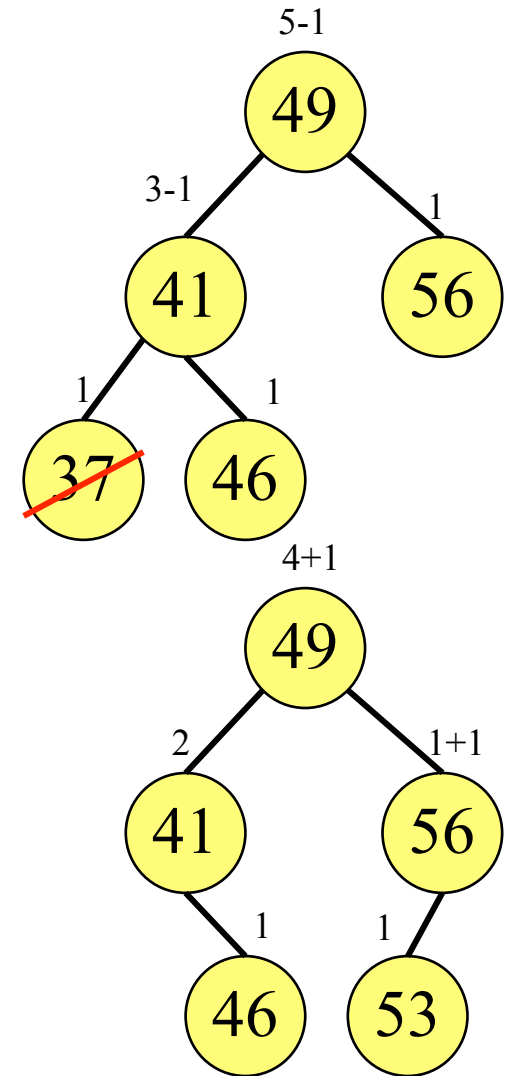
- Each node x has:
 - $\text{key}[x]$
 - Pointers: $\text{left}[x]$, $\text{right}[x]$, $p[x]$
- Property: for any node x :
 - For all nodes y in the **left** subtree of x :
$$\text{key}[y] \leq \text{key}[x]$$
 - For all nodes y in the **right** subtree of x :
$$\text{key}[y] \geq \text{key}[x]$$



height = 3

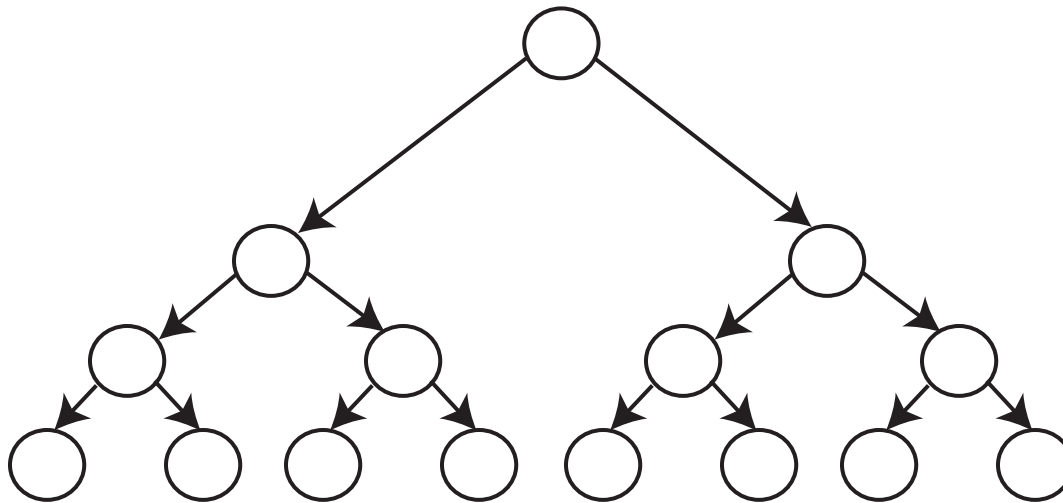
BST Basic Operations

- Find, successor, min, ...
- Remove an element (e.g., 37)
- Insert new element (e.g., 53)
- Delete & insert: $O(h)$,
where h is the height of the tree
- Useful to “augment” a BST (e.g., w/ tree size)



The importance of being balanced

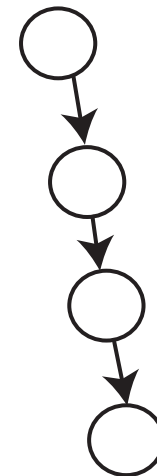
for n nodes:



Perfectly Balanced

$$h = \Theta(\log n)$$

vs.



Path

$$h = \Theta(n)$$

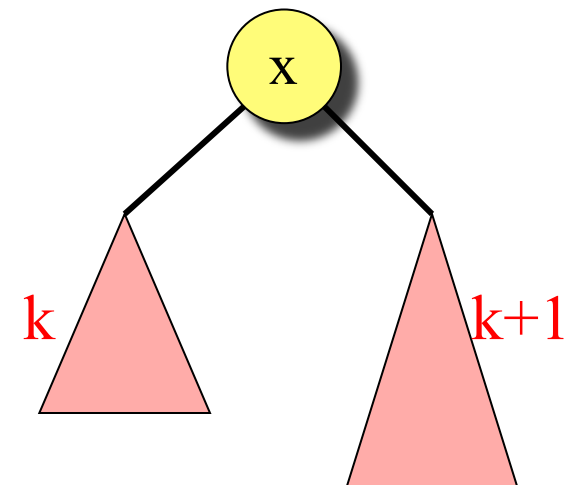
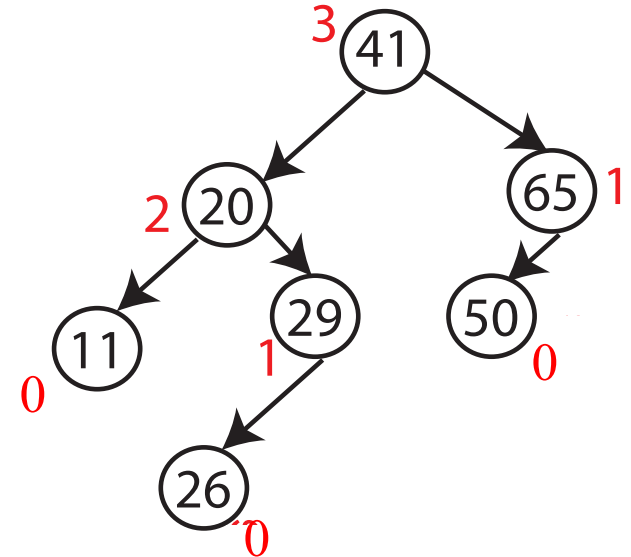
Balanced BST Strategy

- Augment every node with useful INFO
- Define a local invariant on INFO
- Show that invariant guarantees $\Theta(\log n)$ height
- Design algorithms to maintain INFO & invariant

AVL Trees: Definition

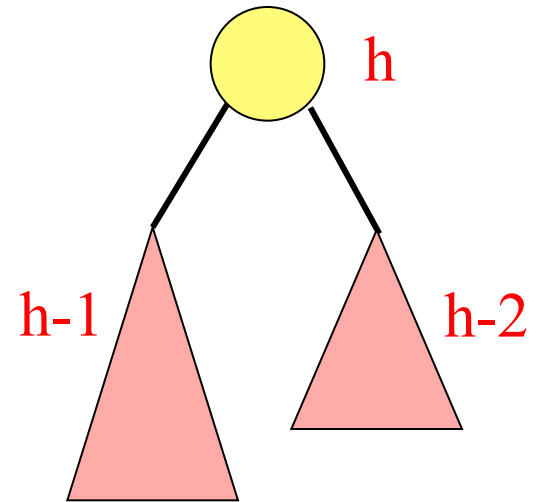
[Adelson-Velskii and Landis'62]

- **INFO:** for every node, store its height (“augmentation”)
 - Leaves have height 0
 - NIL has “height” -1
- **Invariant:** for every node x , the heights of its left child and right child differ by at most 1

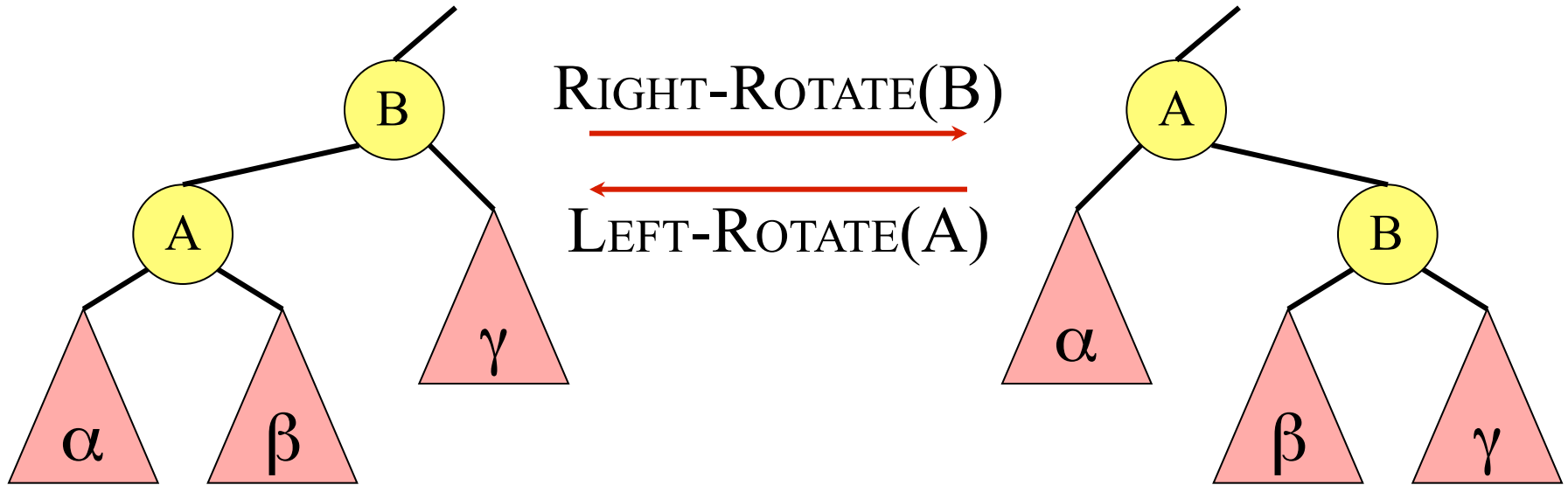


AVL trees have height $\Theta(\log n)$

- Let n_h be the minimum number of nodes of an AVL tree of height h
- We have $n_h \geq 1 + n_{h-1} + n_{h-2}$
 - $\Rightarrow n_h > 2n_{h-2}$
 - $\Rightarrow n_h > 2^{h/2}$
 - $\Rightarrow h < 2 \lg n_h$
- Optimal?

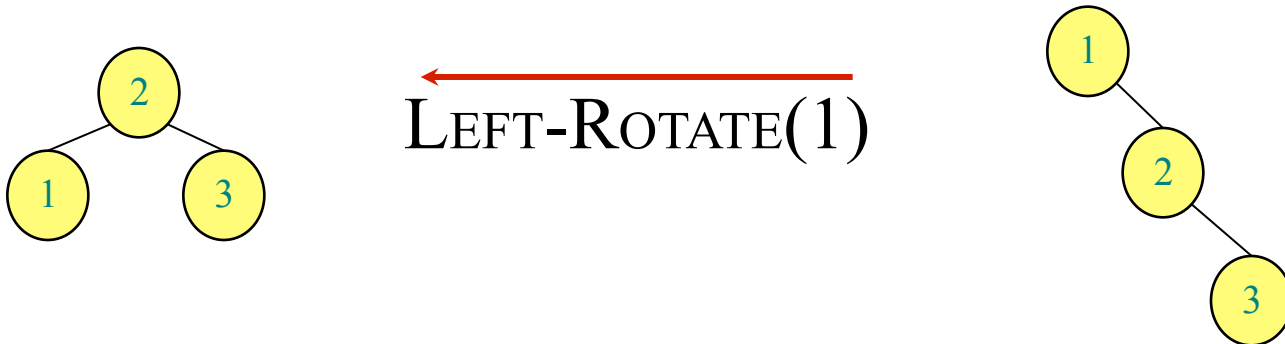


Rotations



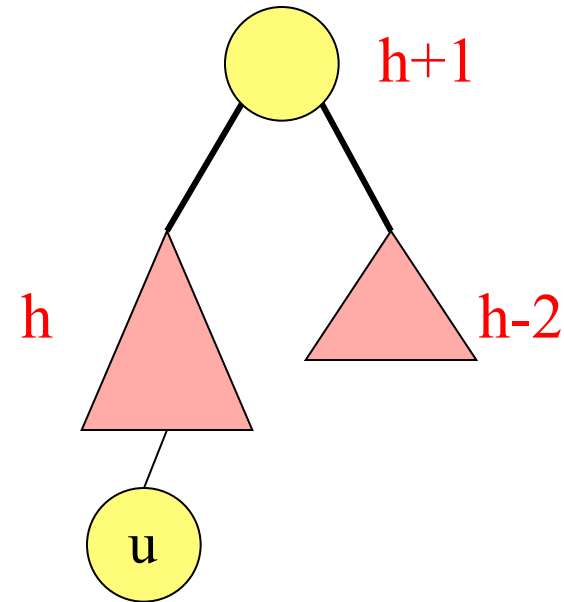
Rotations maintain the inorder ordering of keys:

- $\forall a \in \alpha \quad \forall b \in \beta \quad \forall c \in \gamma: \quad a \leq A \leq b \leq B \leq c.$



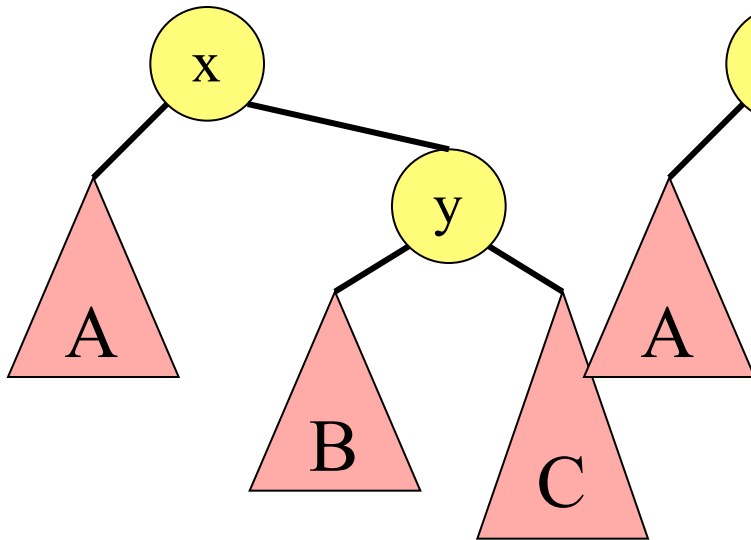
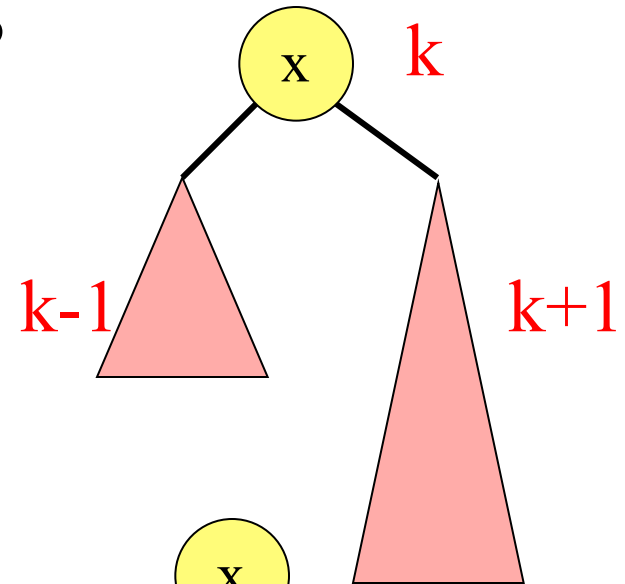
Insertions

- Insert new node u as in the simple BST
 - Can create imbalance
- Work your way up the tree, restoring the balance

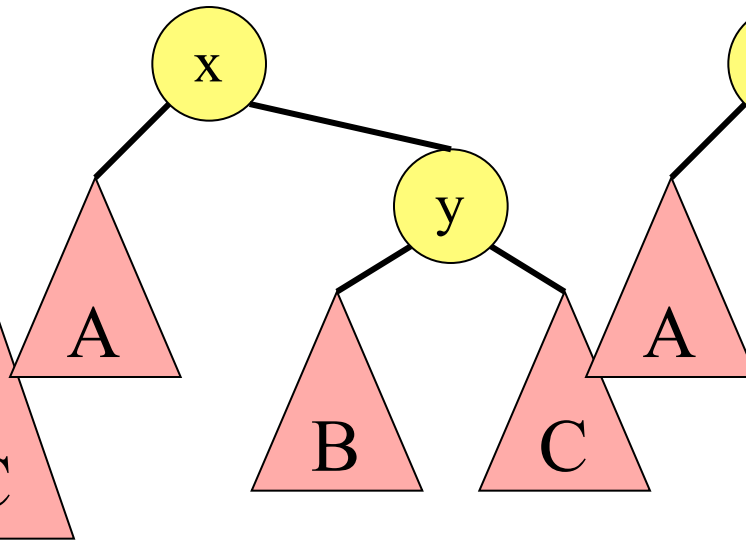


Balancing

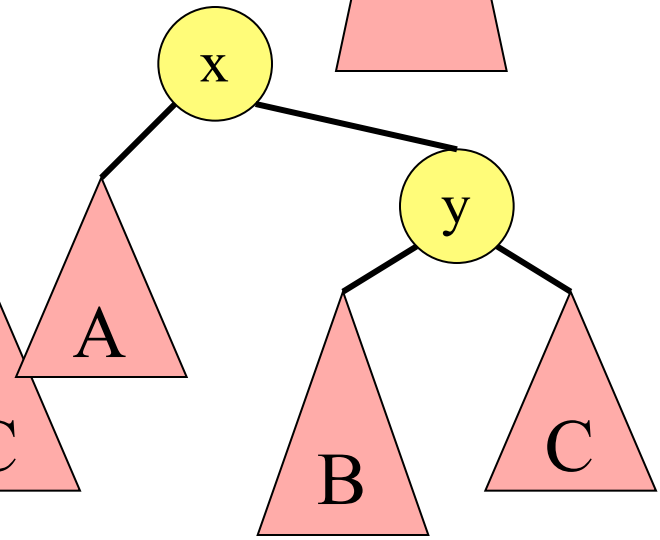
- Let x be the lowest “**violating**” node
- **WLOG** x is “right-heavy”:
Right[x] deeper Left[x]
- 3 Cases (others are symmetric):



1. y right-heavy

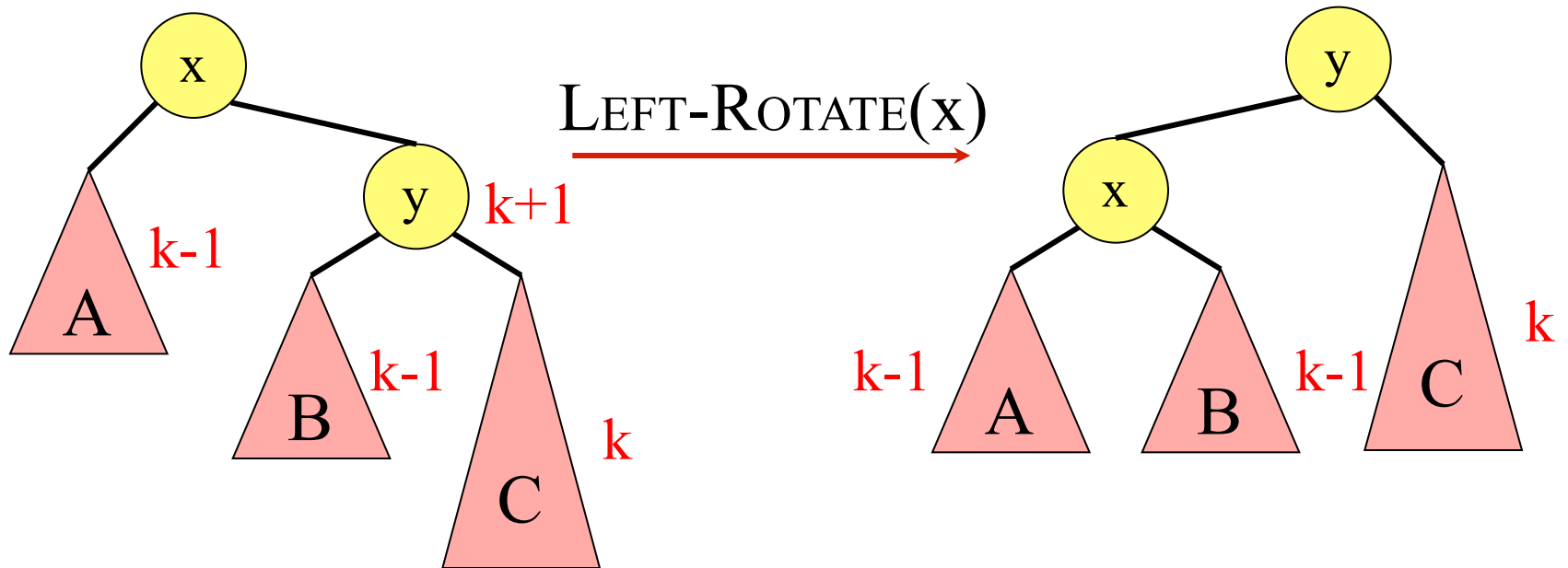


2. y balanced

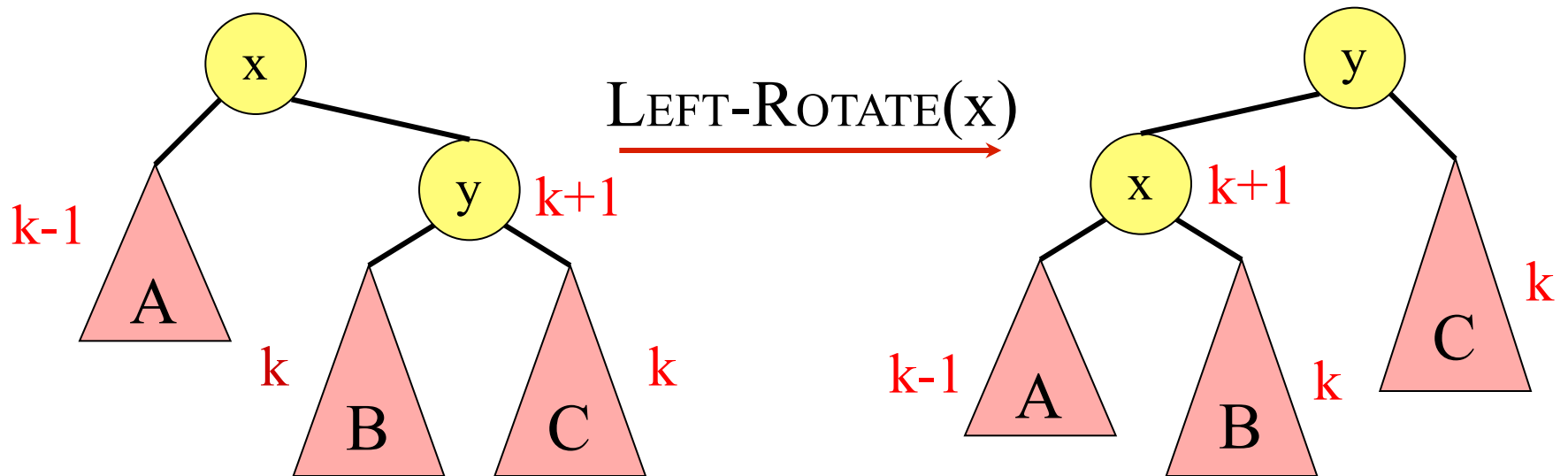


3. y left-heavy

Case 1: y is right-heavy

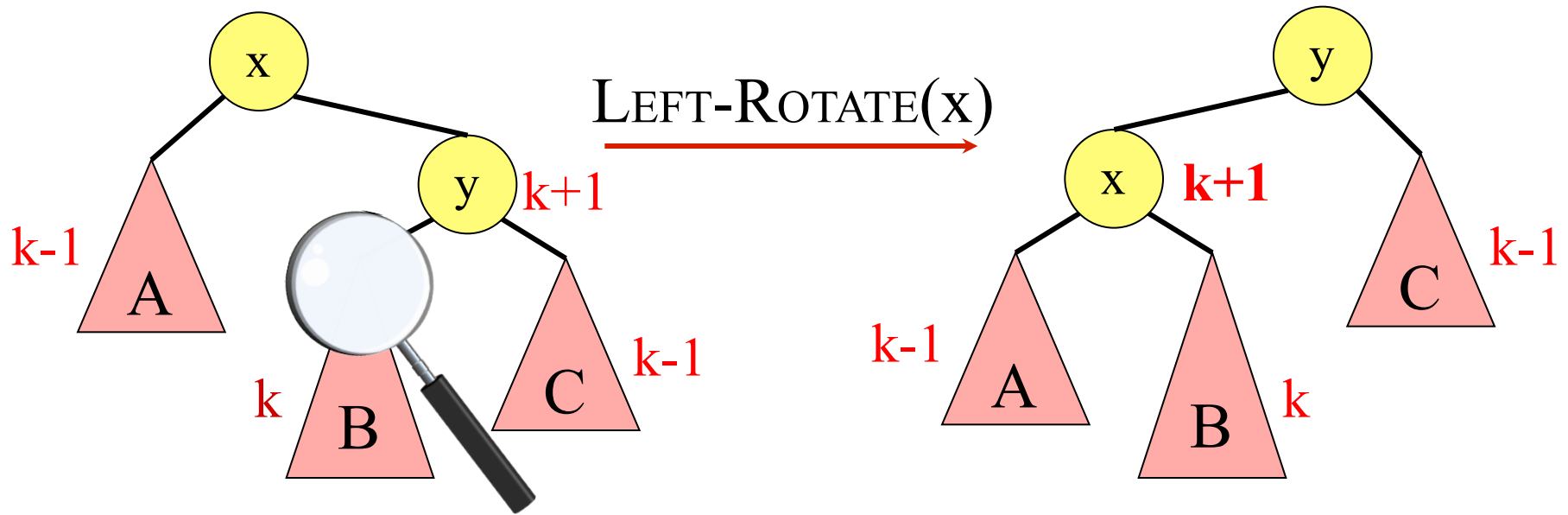


Case 2: y is balanced



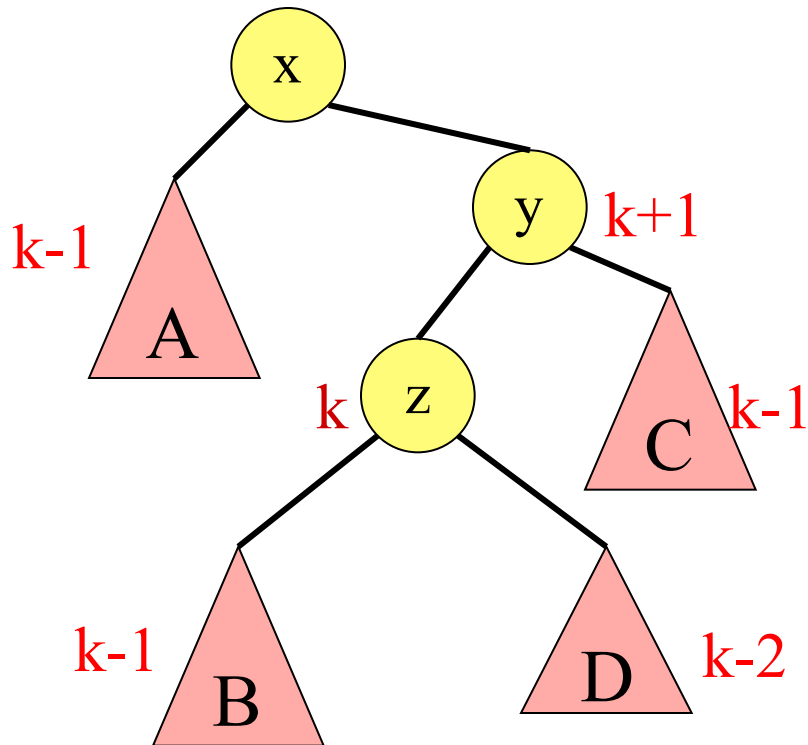
(Same as Case 1)

Case 3: y is left-heavy



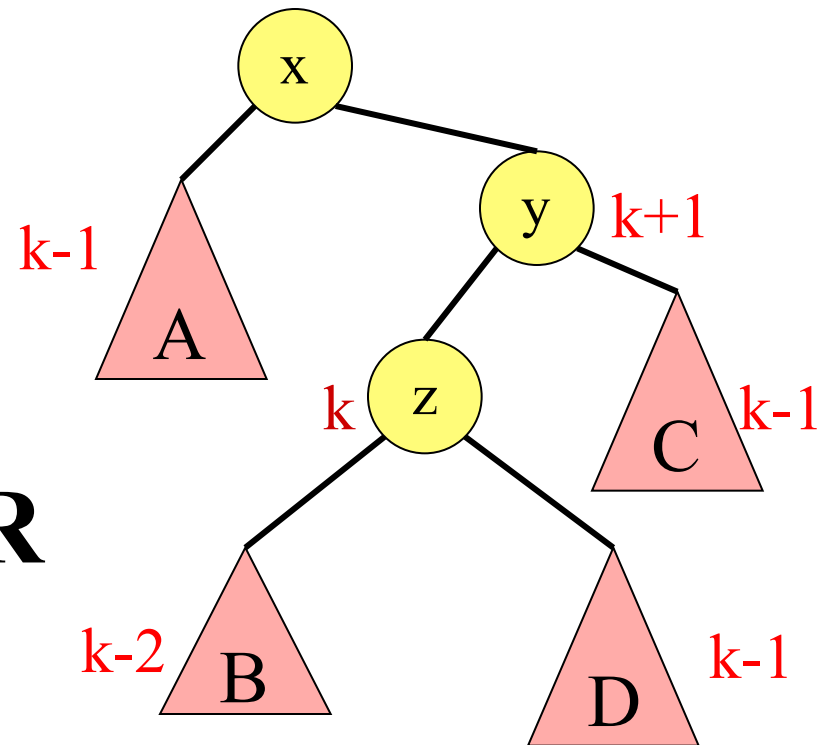
Need to do more ...

Case 3: y is left-heavy



& z is left-heavy

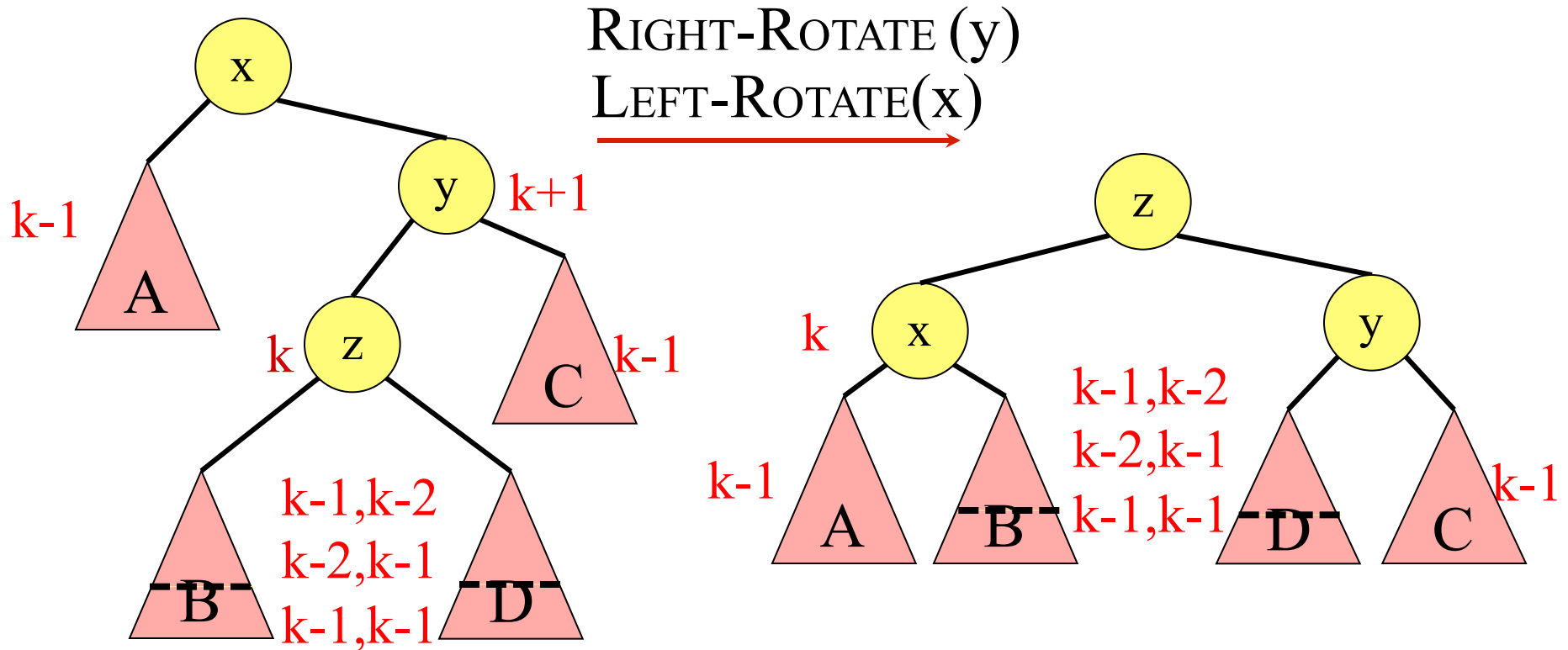
OR



& z is right-heavy

OR ...

Case 3: y is left-heavy



And we are done!

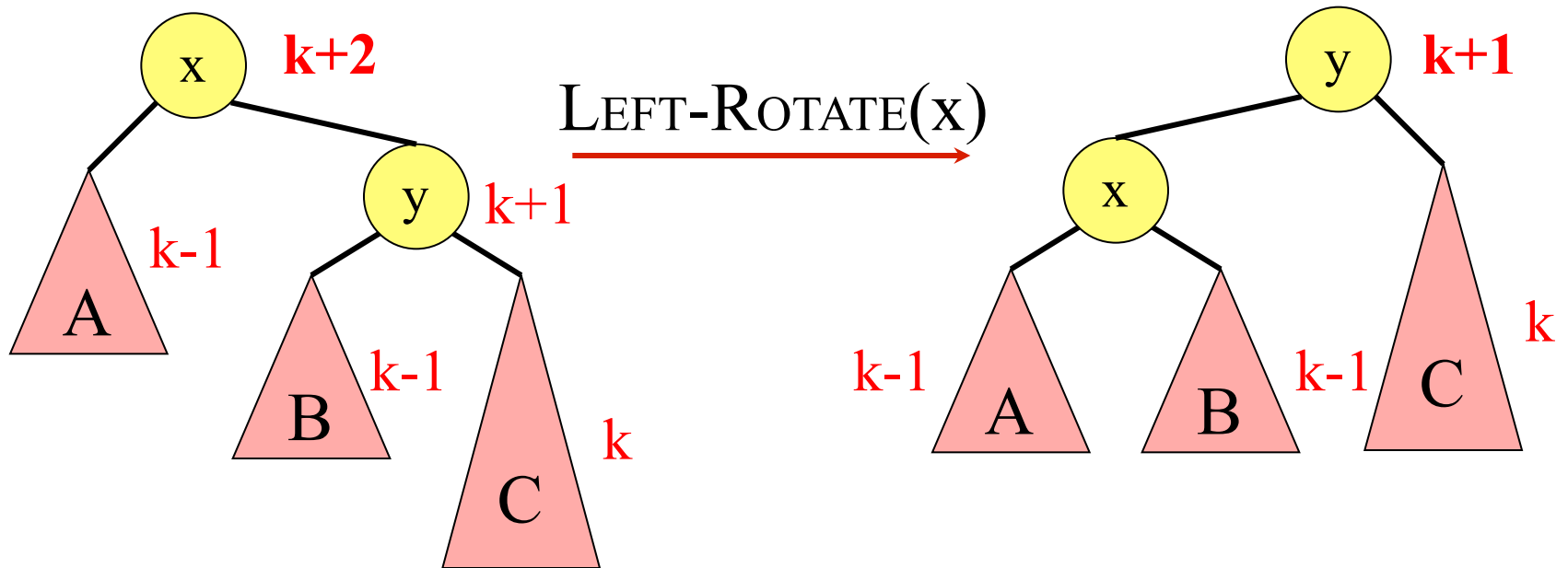
Complexity

Insertion:

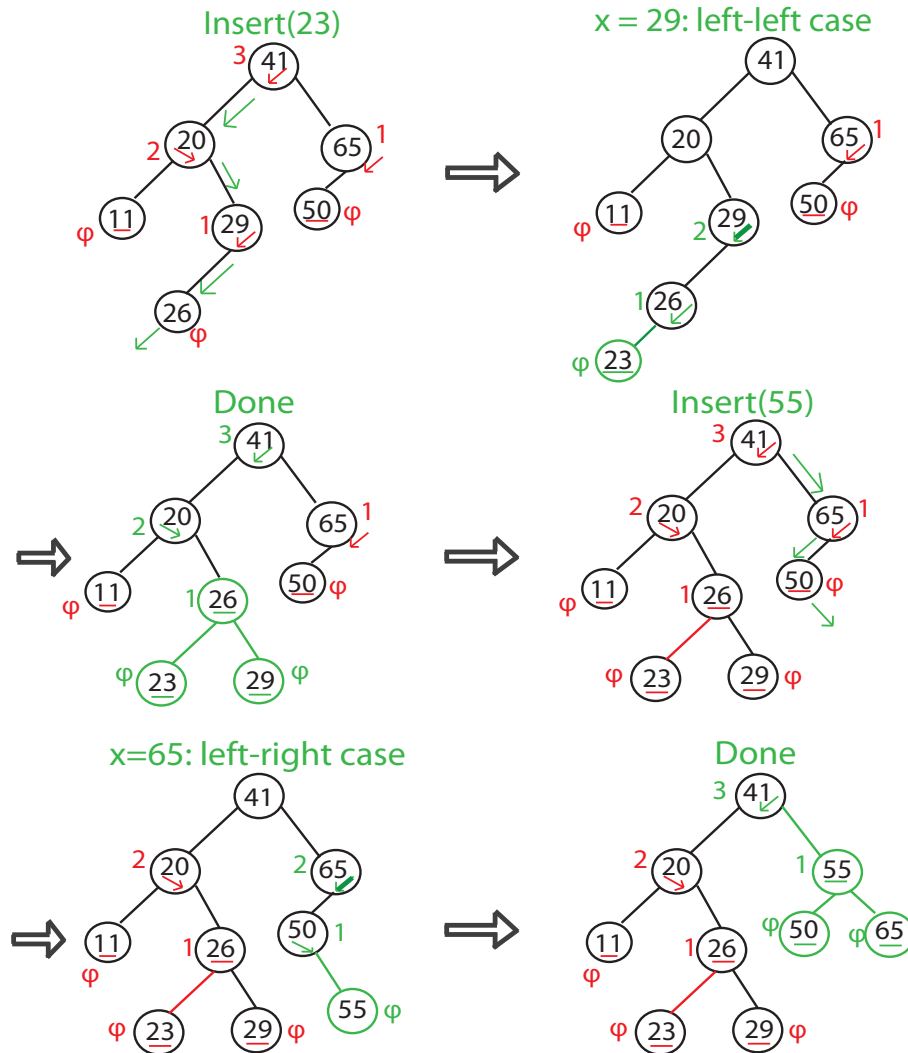
Local rebalancing:

How many local rebalancings after one insertion?

Recall Case 1: y is right-heavy

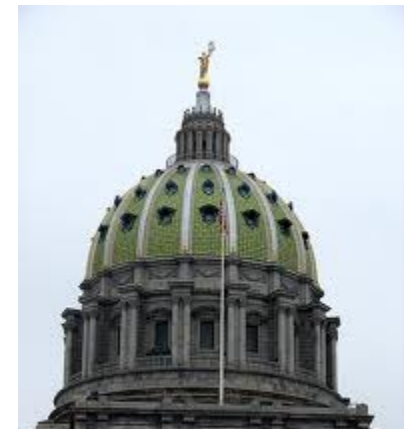


Examples of insert/balancing



Balanced Search Trees ...

- AVL trees (Adelson-Velsii and Landis 1962)
- Red-black trees (see CLRS 13)
- Splay trees (Sleator and Tarjan 1985)
- Scapegoat trees (Galperin and Rivest 1993)
- Treaps (Seidel and Aragon 1996)
-



...

US capitol

Mississippi Arkansas

?

Who invented the Multiplication Algorithm?