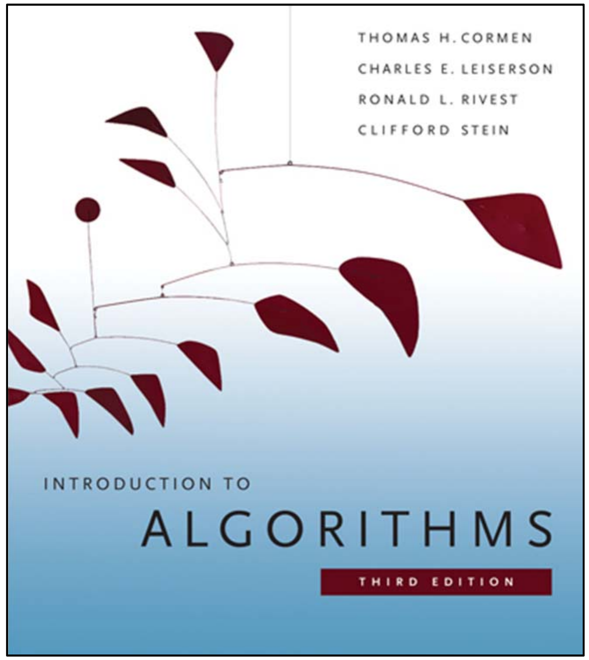
# 6.006

# *Introduction to Algorithms*




# Lecture 24: Geometry

Prof. Erik Demaine

# Today

- Computational geometry
- Line-segment intersection
  - Sweep-line technique
- Closest pair of points
  - Divide & conquer strikes back!

# Motivation: Collision Detection

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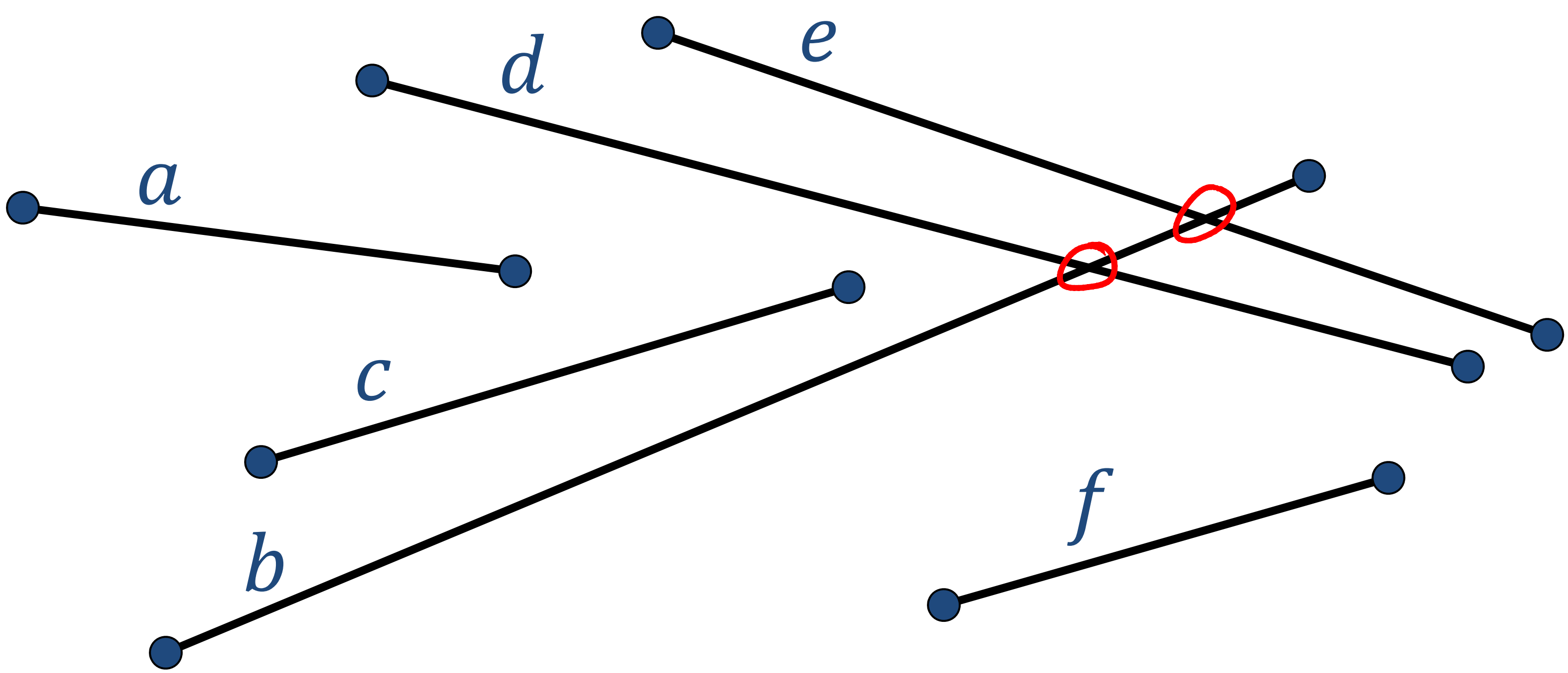
"GTA 4 Carmageddon!" by dot12321
http://www.youtube.com/watch?v=4-620xx7yTo

# Line-Segment Intersection

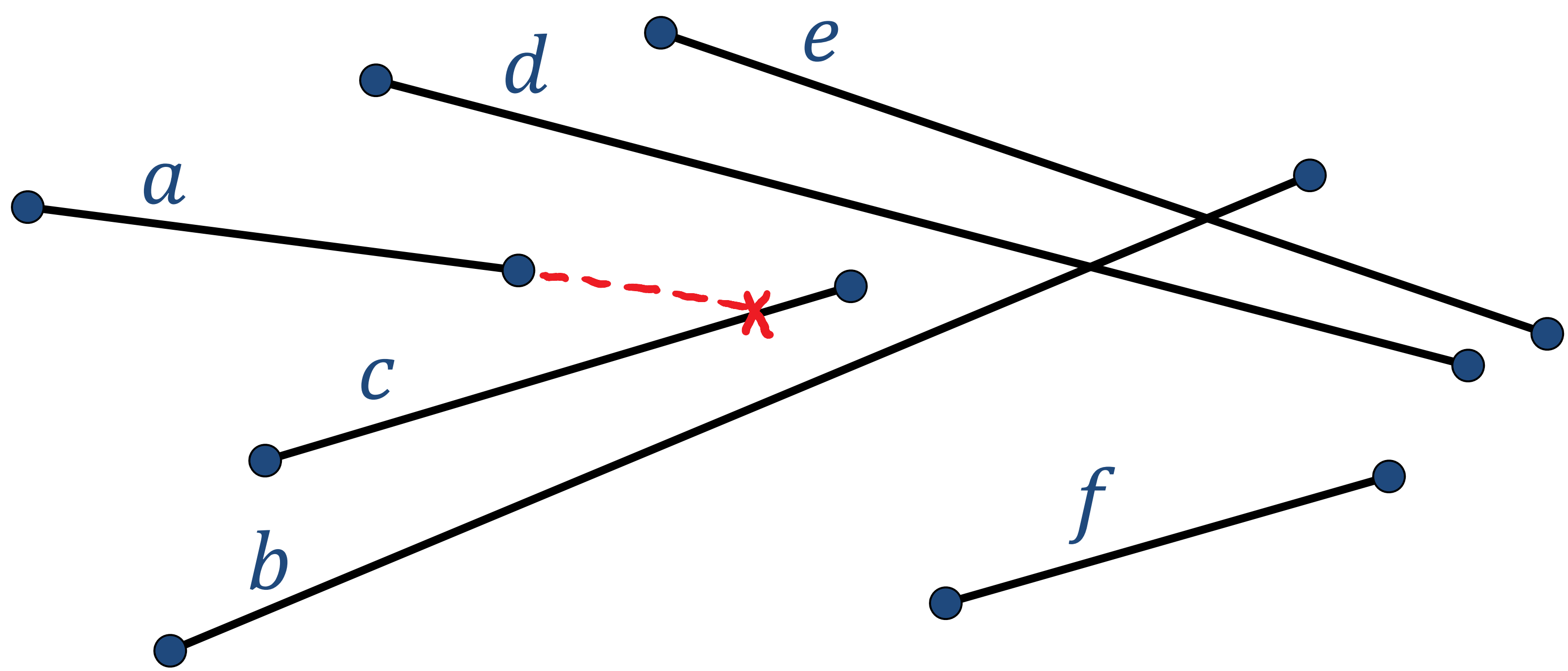
- Input: $n$ line segments in 2D
- Goal: Find the $k$ intersections



# Obvious Algorithm

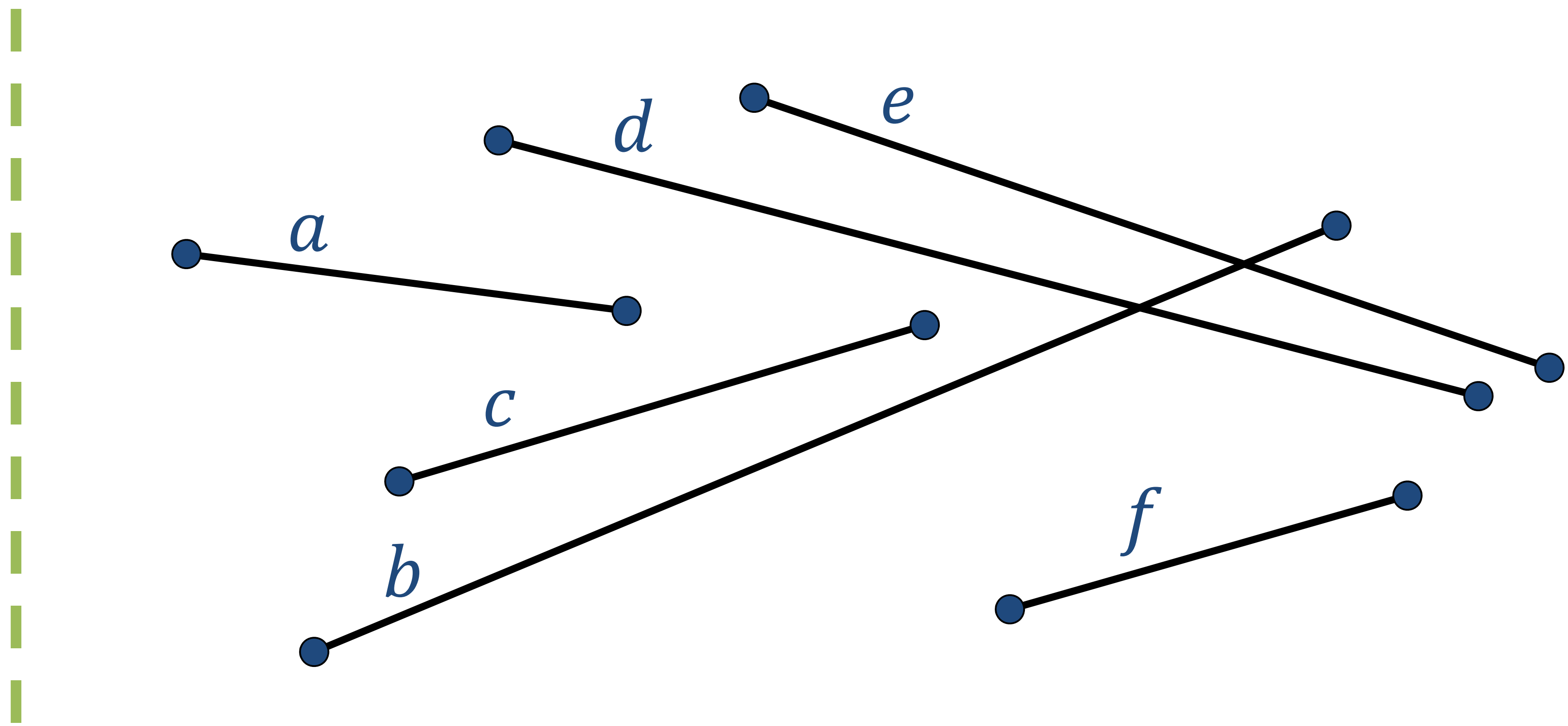
for every pair $(\ell, \ell')$ of line segments:
    check for intersection

$\Big\}\ O(n^2)$

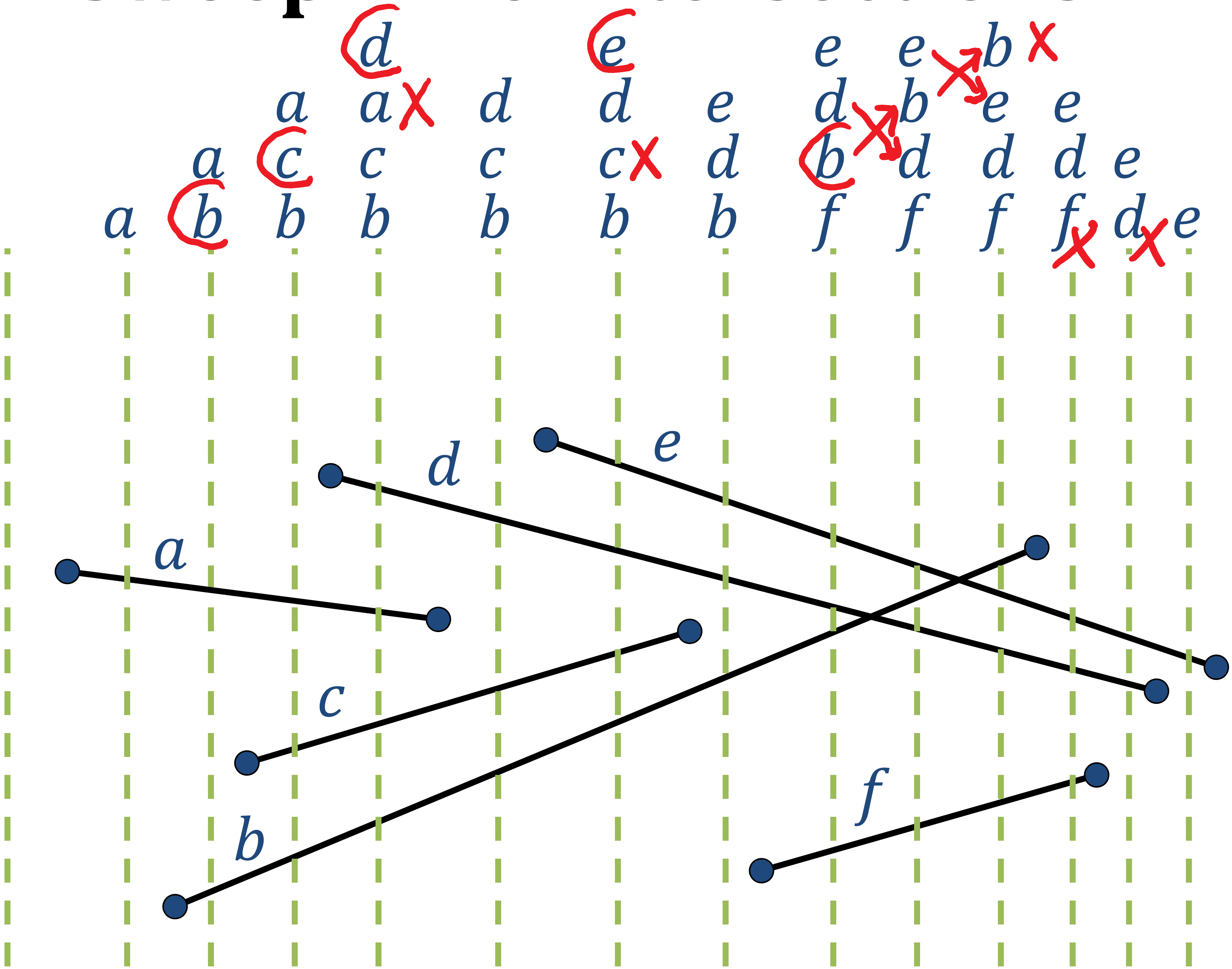


# Sweep-Line Technique

- <u>Idea:</u> Sweep a vertical line from left to right
- Maintain intersection of line with input

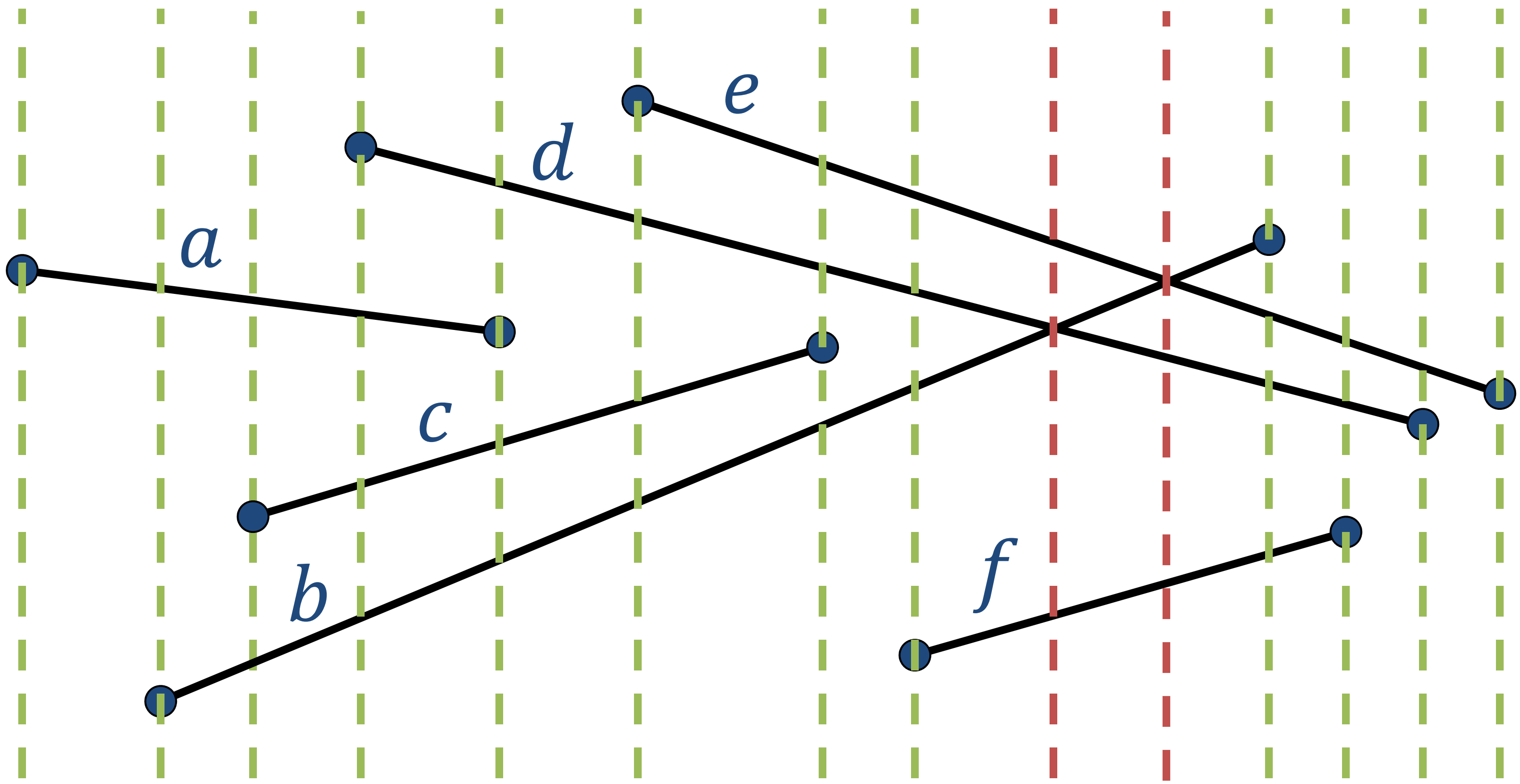


# Sweep-Line Intersections

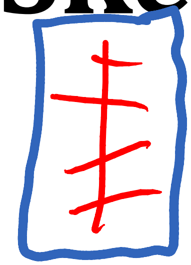


# Sweep-Line Events

- Discretize continuous motion of sweep line
- ***Events*** when intersection changes
  - Segment endpoints
  - Intersections



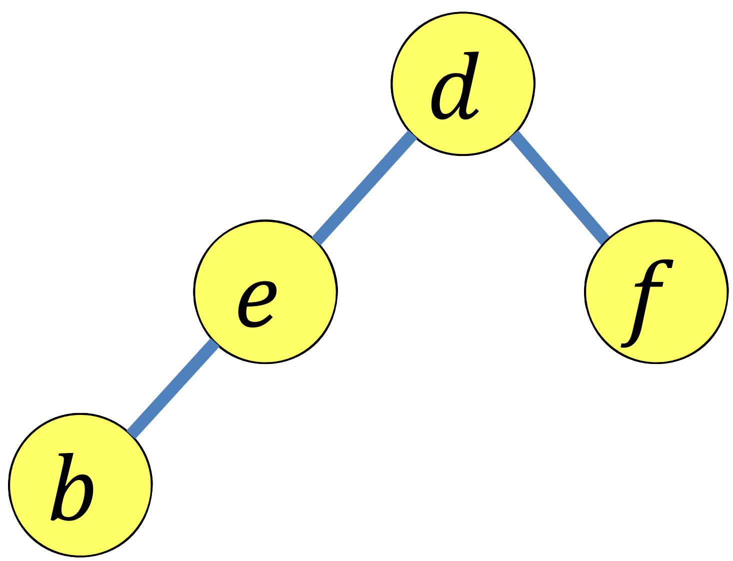
# Sweep-Line Algorithm Sketch

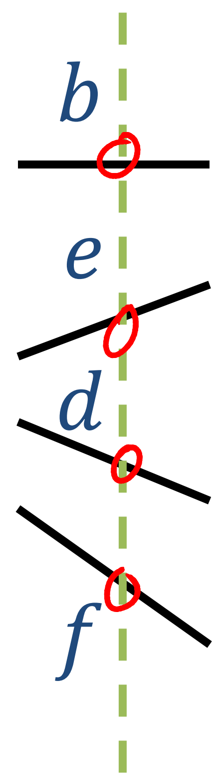
- Maintain sweep-line intersection 
- Maintain **priority queue** of (possible) event ***times*** ($= x$ coordinates of sweep line)
- Until queue is empty:
  - Delete minimum event time $t$ from priority queue
  - Update sweep-line intersection from $< t$ to $> t$
  - Update possible event times in priority queue

*“Discrete-event simulation”*

# Intersection Data Structure

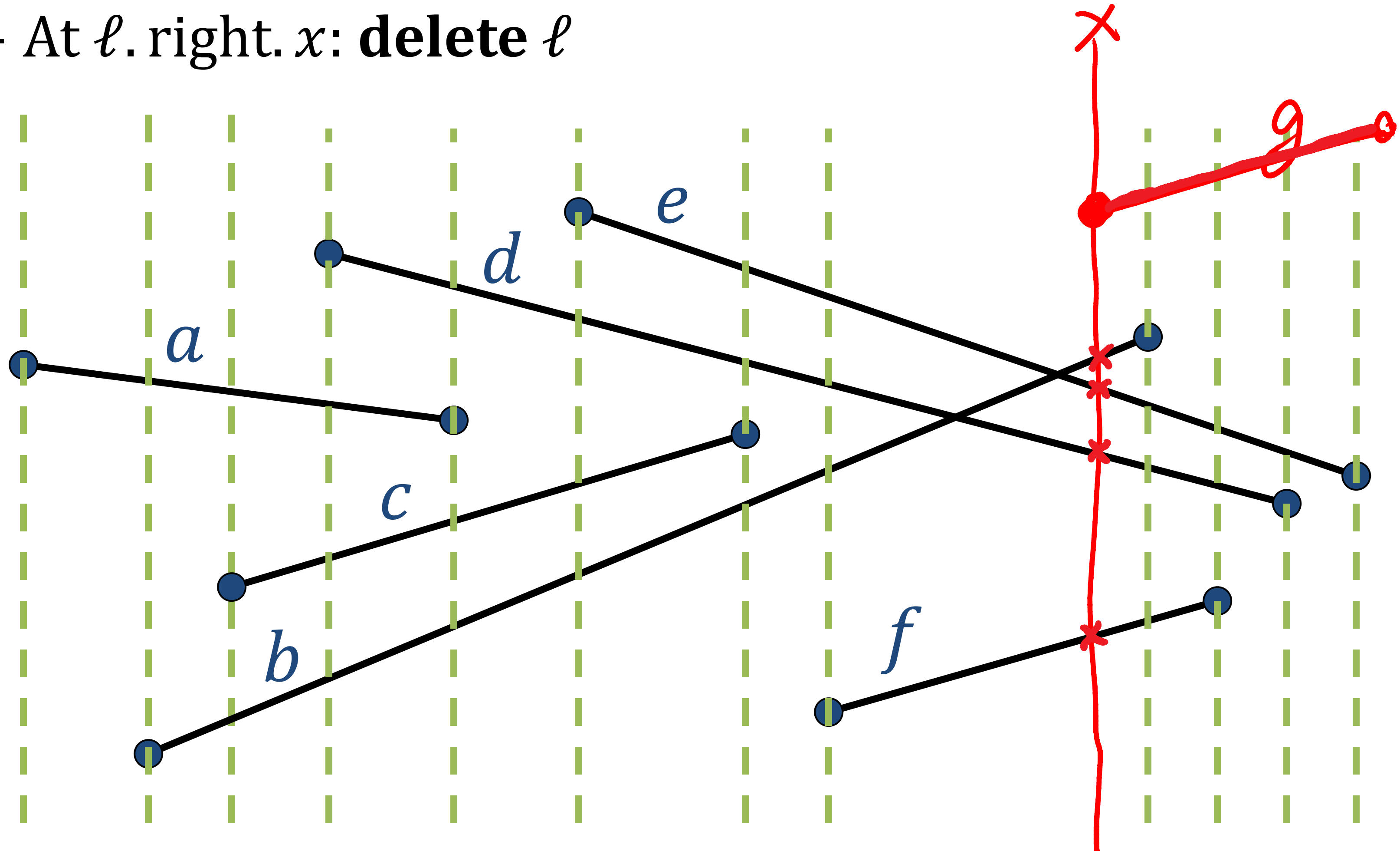
- Balanced binary search tree (e.g., AVL tree) to store sorted ($y$) order of intersections



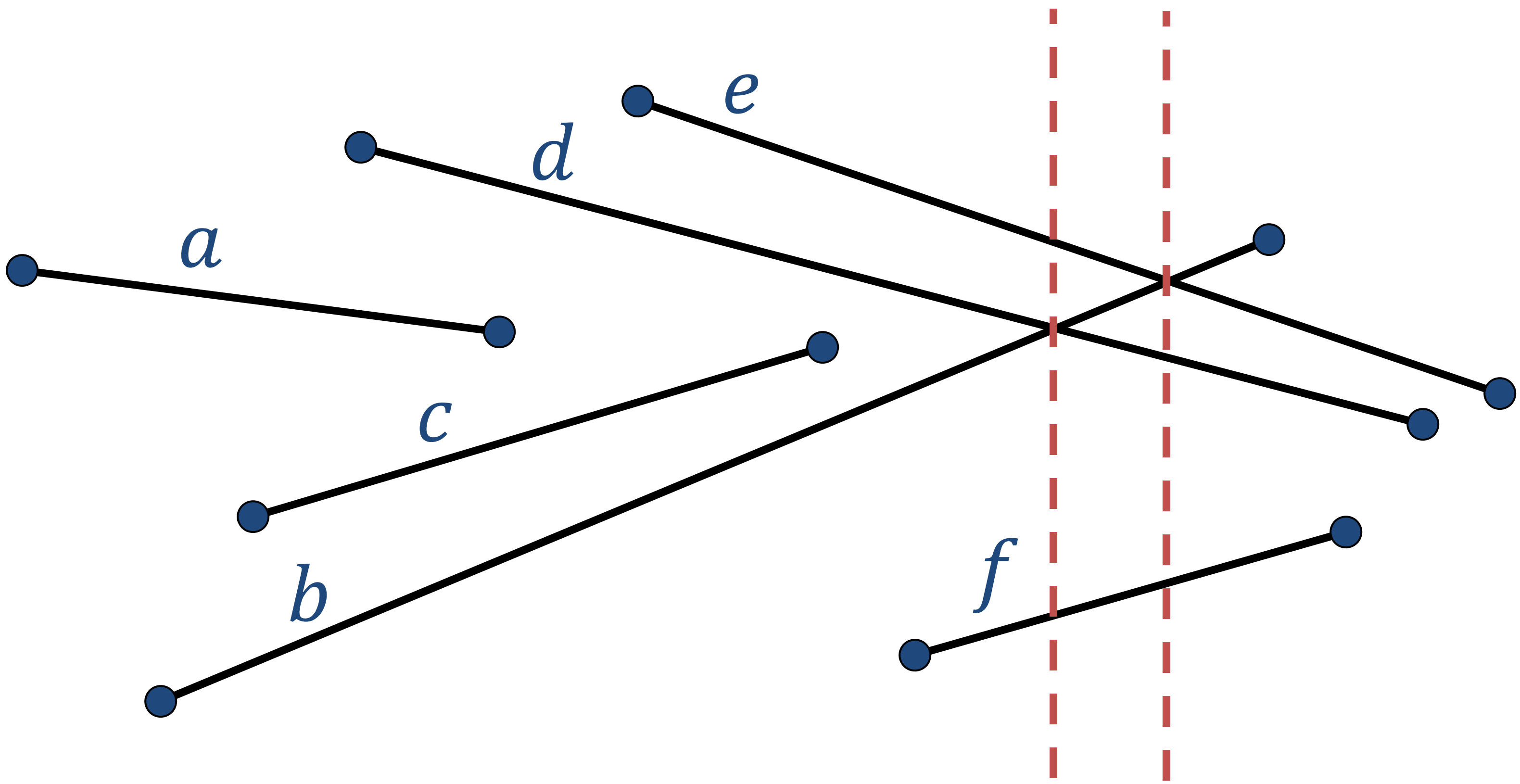
# Endpoint Events

- For each line segment $\ell = (\ell.\text{left}, \ell.\text{right})$:
  - At $\ell.\text{left}.x$: **insert** $\ell$ (binary search for neighbors then)
  - At $\ell.\text{right}.x$: **delete** $\ell$



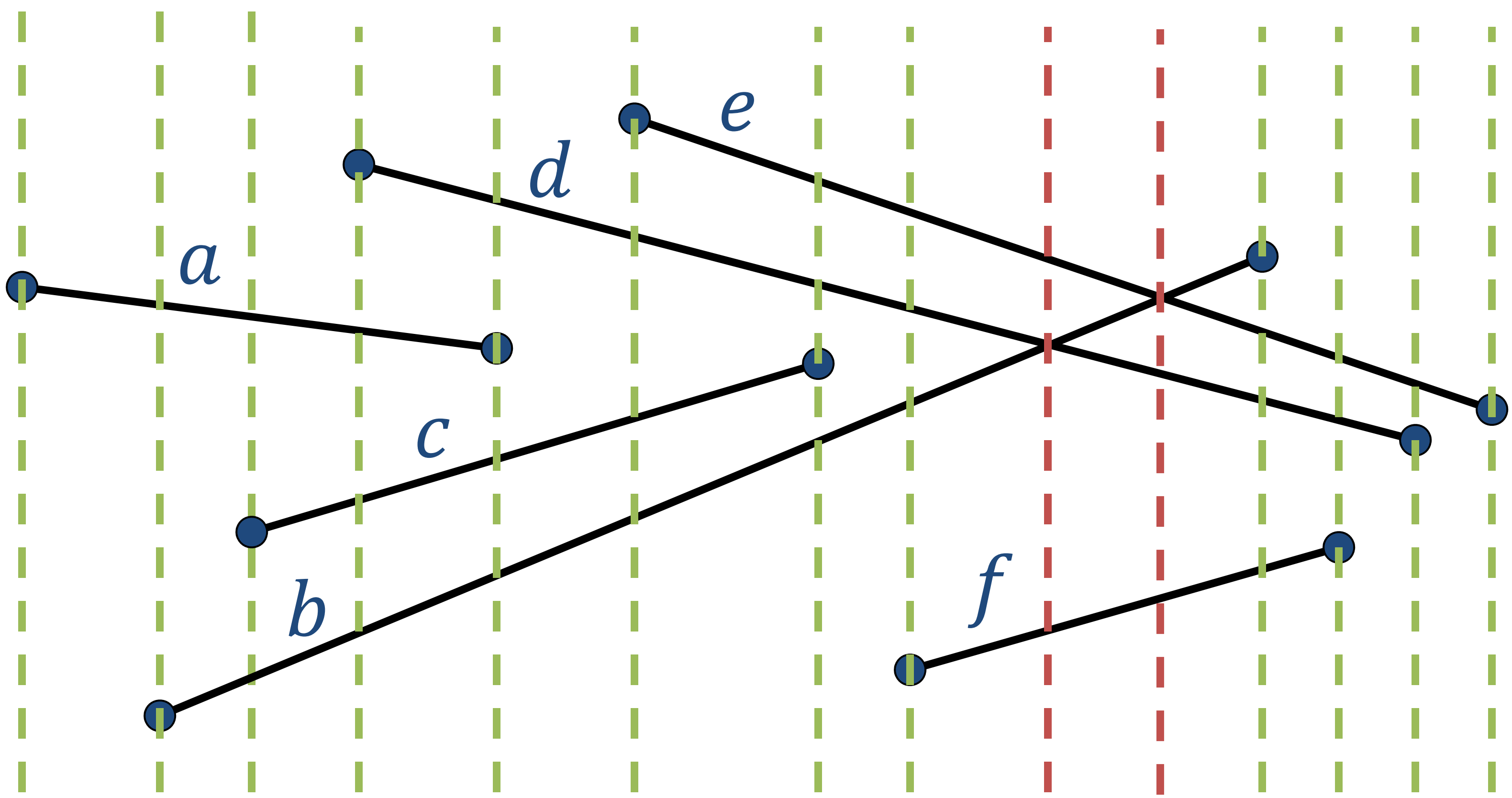
# Intersection Events?

- How to know when have intersection events?
- This is the whole problem!



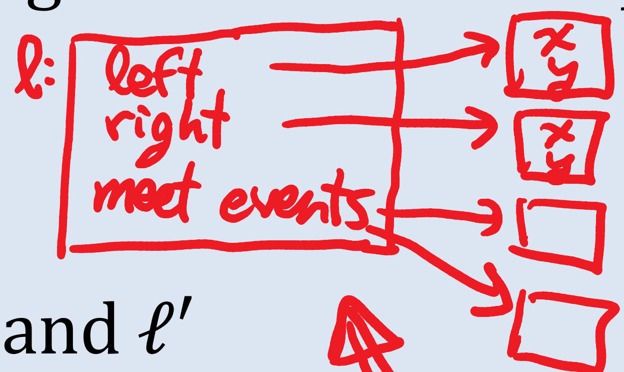
# Intersection Events

- Compute when neighboring segments would intersect, if nothing changes meanwhile
- If such an event is next, then it really happens



# Sweep-Line Algorithm

$T$ = empty AVL tree
$Q$ = Build−Heap($\{\ell.\text{left}.x, \ell.\text{right}.x$ for segment $\ell\}$)
while $Q$ is not empty:
    event = $Q$.delete−min()
    if event is $\ell.\text{left}.x$:
        insert $\ell$ into $T$ (binary searching with $x = \ell.\text{left}.x$)
    elif event is $\ell.\text{right}.x$:
        delete $\ell$ from $T$
    elif event is meet($\ell, \ell'$):
        report intersection between $\ell$ and $\ell'$
        swap contents of nodes for $\ell$ and $\ell'$ in $T$
    for each node $\ell$ changed or neighboring changed in $T$:
        delete all meet events involving $\ell$ from $Q$
        add meet($\ell$, $T$.pred($\ell$)) & meet($\ell$, $T$.succ($\ell$)) to $Q$




$O(1)$

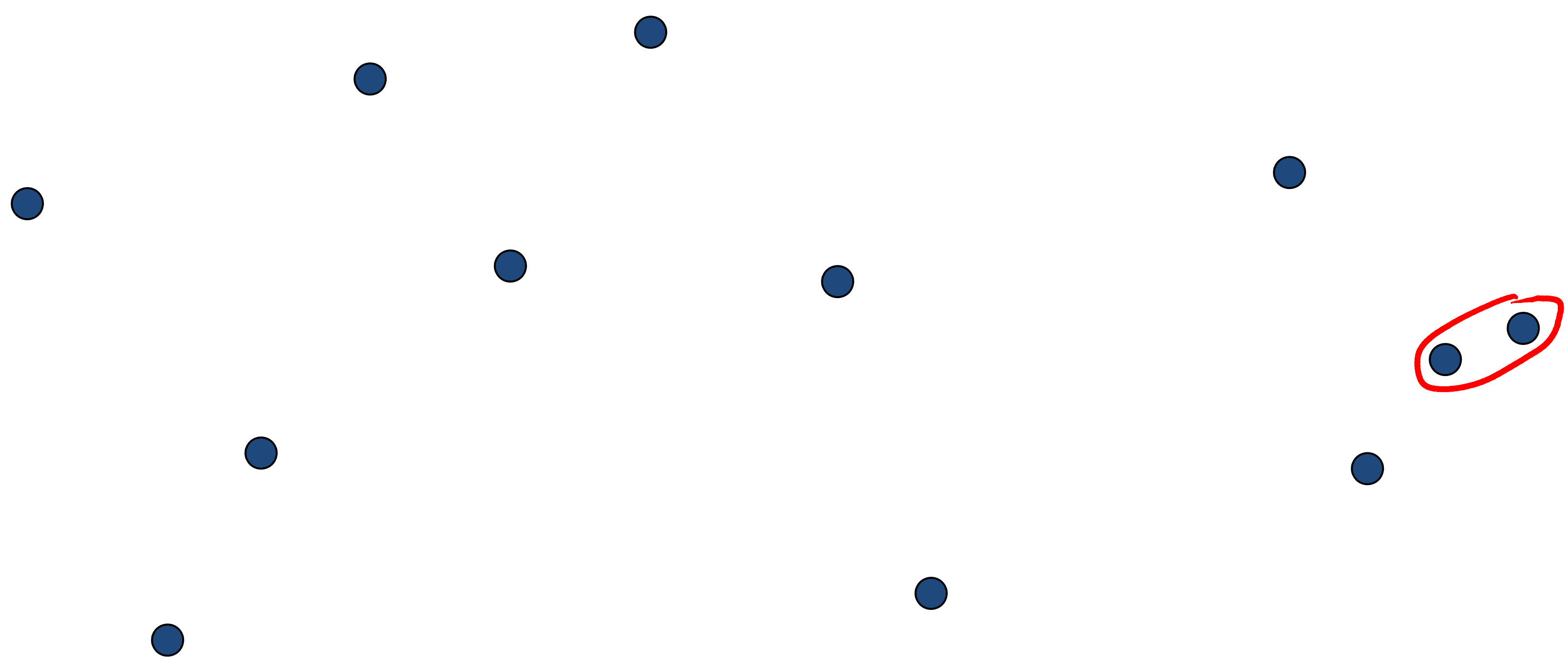
# Sweep-Line Analysis

- Running time = $O(\#\text{ events} \cdot \lg(\#\text{ events}))$
- Number of endpoint events = $2n$
- Number of meet events = number $k$ of intersections = size of output
- Running time = $O((n+k)\lg n)$

  $\lg(n+k) \le \lg(n+n^2) = O(\lg n)$
- ***Output sensitive algorithm:*** running time depends on size of output
- Best algorithm runs in $O(n \lg n + k)$ time

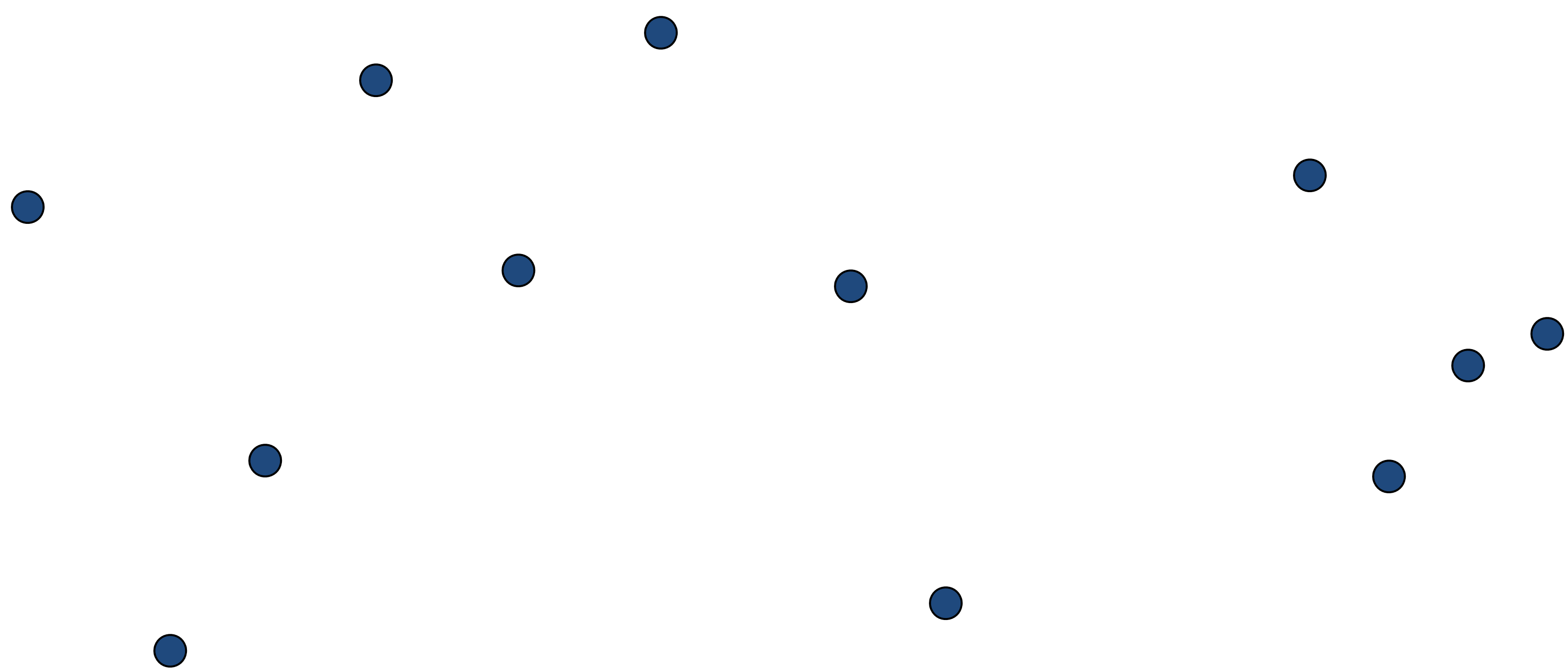
# Closest Pair of Points

- Input: $n$ points in 2D
- Goal: find two closest points



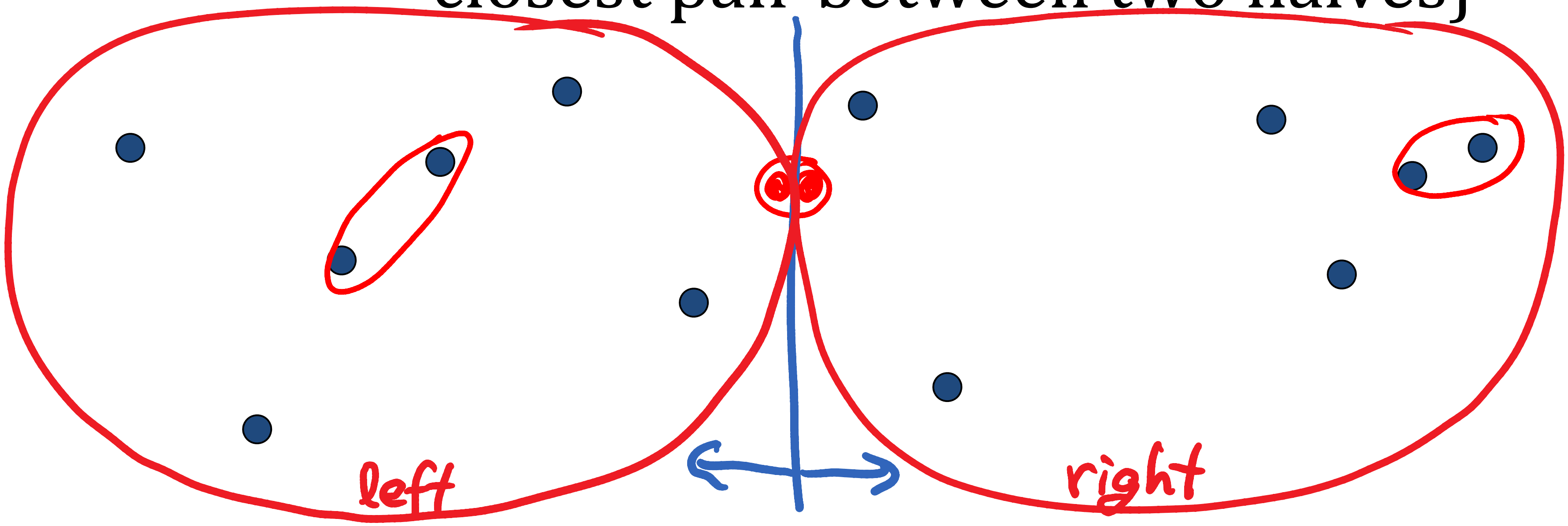
# Obvious Algorithm

$$\min(\text{distance}(p, q) \\ \quad \text{for every pair } (p, q) \text{ of points}) \quad \Big\} O(n^2)$$



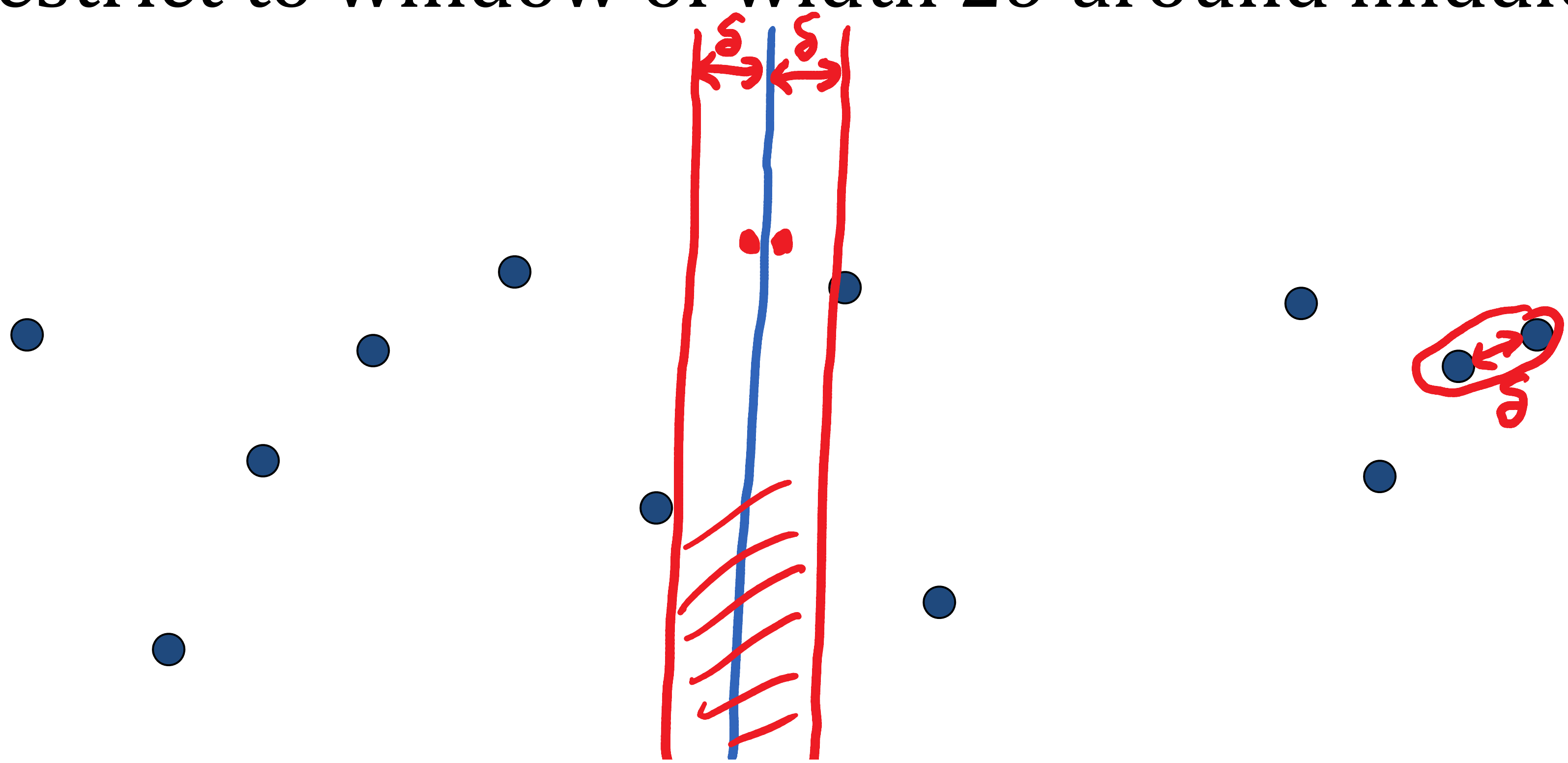
# Divide-and-Conquer Idea

- Initially sort points by $x$ coordinate
- Split points into left half and right half
- Recurse on each half: find closest pair
- return min{closest pair in each half,
  closest pair between two halves}



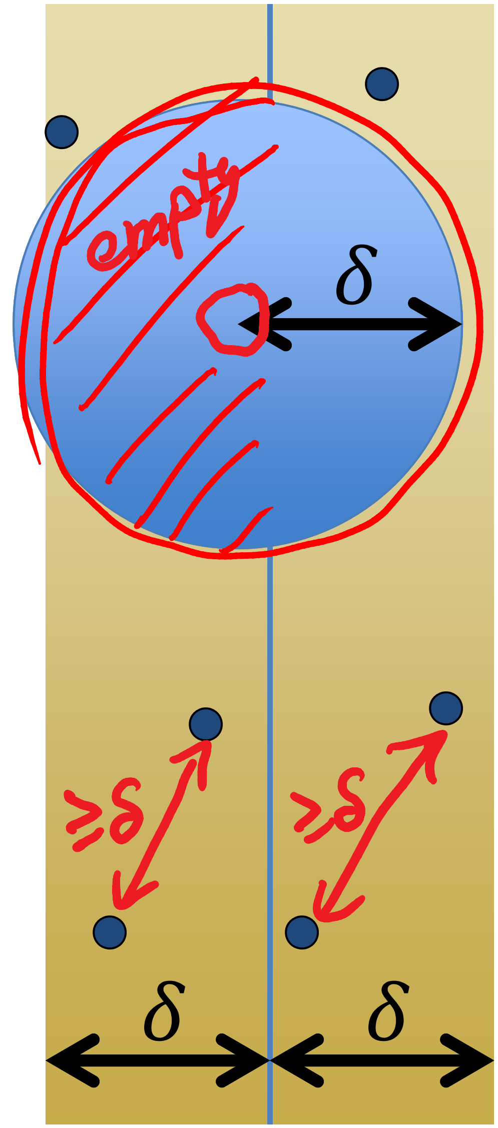
# Closest Pair Between Two Halves?

- Let $\delta$ = min{closest pair in each half}
- Only interested in pairs with distance $< \delta$
- Restrict to window of width $2\delta$ around middle

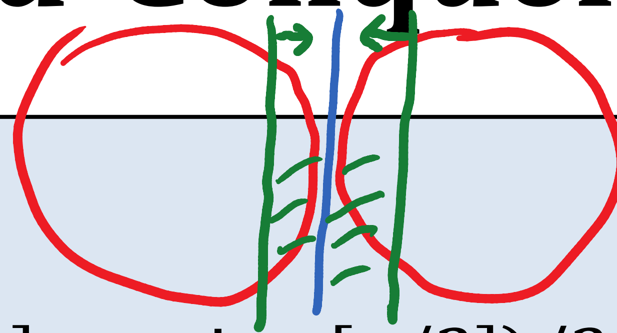


# Closest Pair Between Two Halves

- For each left point, interested in points on right within distance $\delta$
- Points on right side can't be within $\delta$ of each other
- So at most three right points to consider for each left point
  - Ditto for each right point
- Can compute in $O(n)$ time by merging two sorted arrays



# Divide-and-Conquer



presort points by $x$
def closest−pair(points):
    middle $=$ (points$[n/2-1]$ + points$[n/2]$)$/2$    $O(1)$
    $\delta =$ min{closest−pair(points$[:n/2]$) ,
            closest−pair(points$[n/2:]$)}    $2T(\frac{n}{2})$
    sort points[points.succ(middle $-\ \delta$)$: n/2$] by $y$
    sort points$[n/2:$ points.pred(middle $+\ \delta$)] by $y$    $O(n \lg n)$
    merge and find closest pair between two lists    $O(n)$
    return min{$\delta$, closest distance from merge}

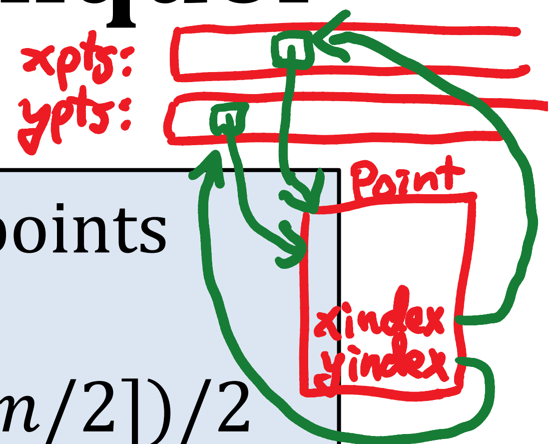
$$n \lg n$$

$$\frac{n}{2} \lg \frac{n}{2} + \frac{n}{2} \lg \frac{n}{2} = n \lg \frac{n}{2} = \boxed{n \lg n} - n$$

$$T(n) = 2T(\tfrac{n}{2}) + O(n \lg n)$$
$$= O(n \lg^2 n)$$

# Faster Divide-and-Conquer

[Shamos & Hoey 1975]




presort points by $x$ and $y$, and cross link points

def closest−pair(xpoints, ypoints):

  $\text{middle} = (\text{xpoints}[n/2-1] + \text{xpoints}[n/2])/2$

  $\delta = \min\{\text{closest–pair}(\text{xpoints}[:n/2], \rightarrow\text{ypoints}),$

    $\text{closest–pair}(\text{xpoints}[n/2:], \rightarrow\text{ypoints})\}$ $\quad 2T(\frac{n}{2})$

  $\text{xpoints}[\text{xpoints.succ}(\text{middle} - \delta) : n/2]$

  $\text{xpoints}[n/2 : \text{xpoints.pred}(\text{middle} + \delta)]$ $\quad O(n)$

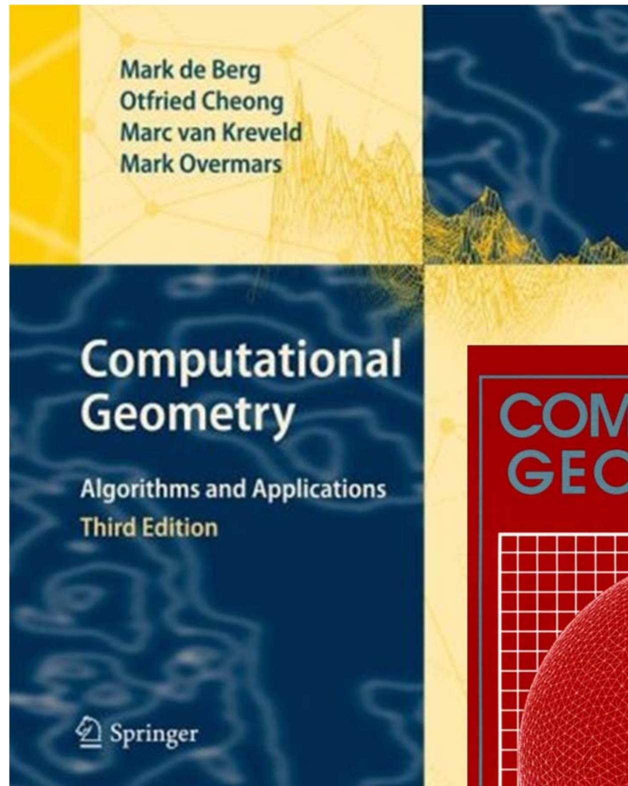
  map to ypoints & find closest pair between lists $\quad O(n)$

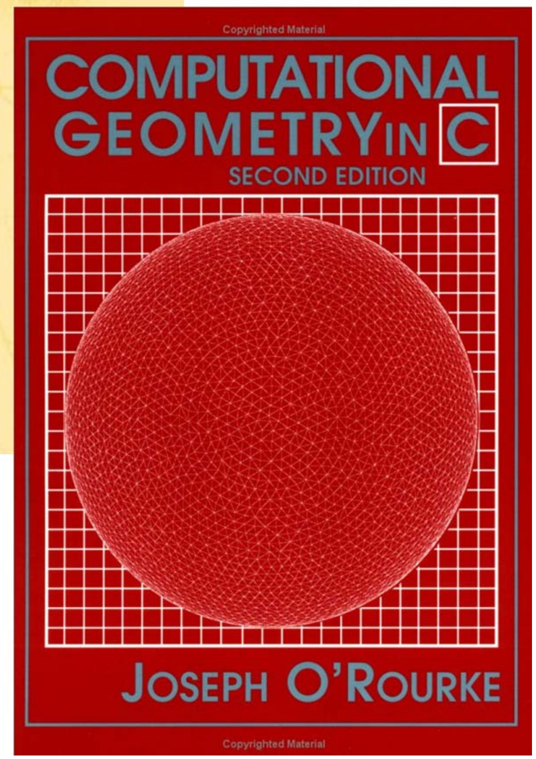
  return $\min\{\delta$, closest distance from merge$\}$

$$T(n) = 2T(\tfrac{n}{2}) + O(n)$$
$$= O(n \lg n)$$

# Other Computational Geometry Problems

- Convex hull
- Voronoi diagram
- Triangulation
- Point location
- Range searching
- Motion planning
- …




**6.850: Geometric Computing**