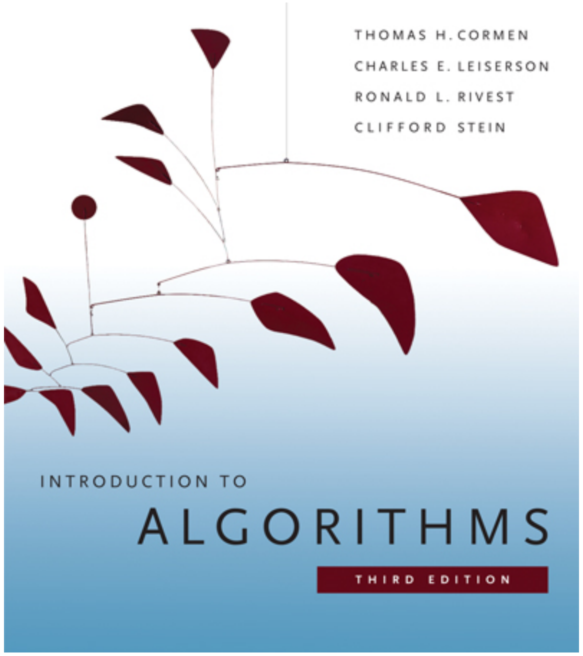
6.006- *Introduction to Algorithms*



Lecture 9

Prof. Piotr Indyk

Menu

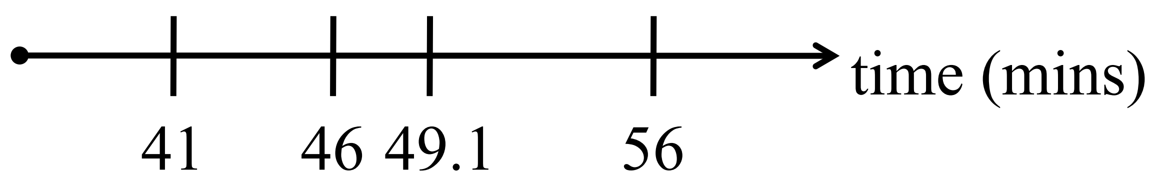
- Priority Queues
- Heaps
- Heapsort

Priority Queue

A data structure implementing a set S of elements, each associated with a key, supporting the following operations:

- insert(S, x) : insert element x into set S
- max(S) : return element of S with largest key
- extract_max(S) : return element of S with largest key and remove it from S
- increase_key(S, x, k) : increase the value of element x’ s key to new value k (assumed to be as large as current value)

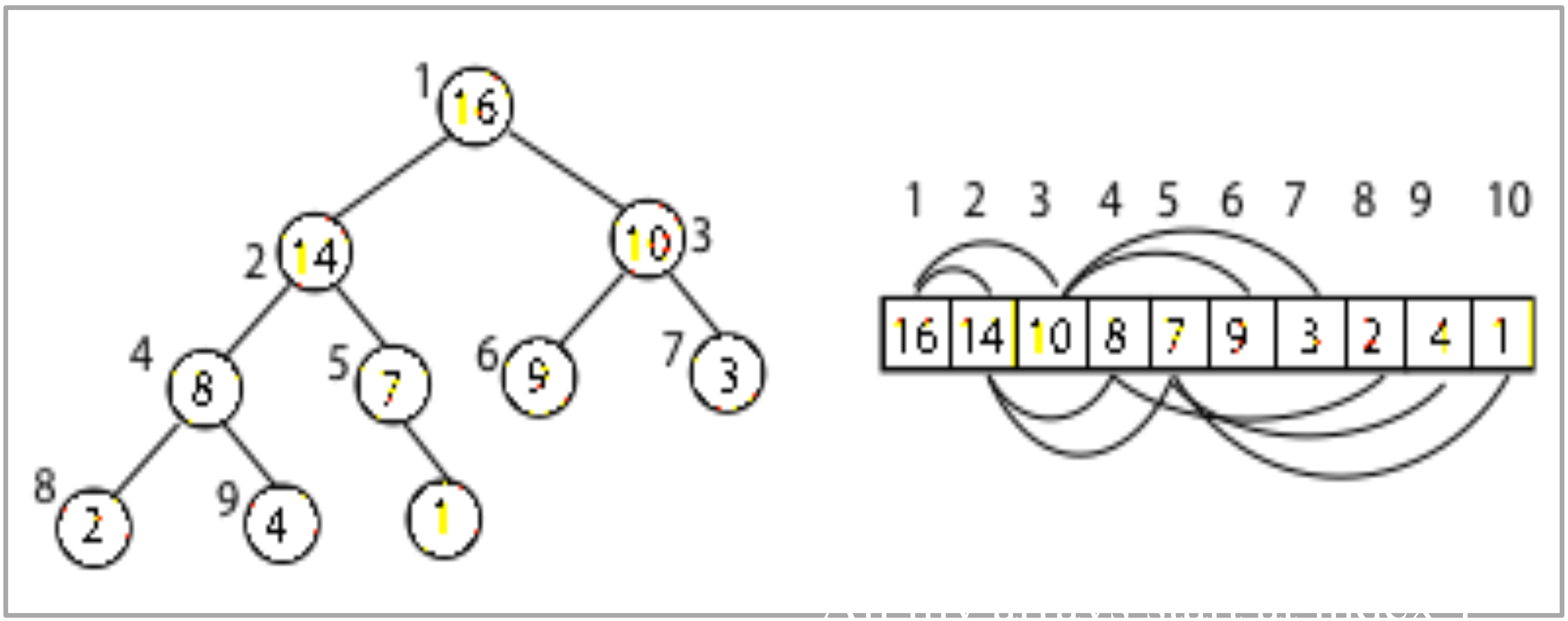
Lecture 3:



Heap

- Implementation of a priority queue (more efficient than BST)
- An array, visualized as a nearly complete binary tree
- Max Heap Property: The key of a node is $\geq$ than the keys of its children

  (Min Heap defined analogously)



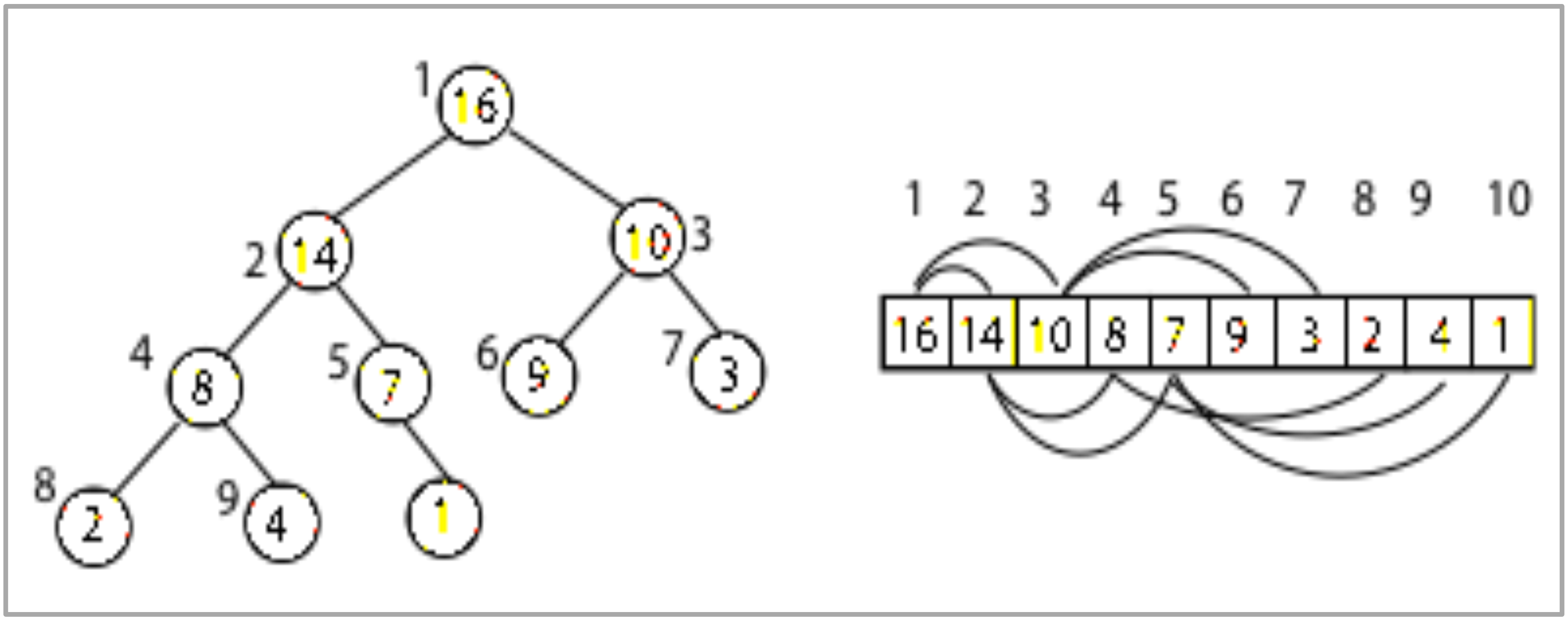
Heap as a Tree

root of tree: first element in the array, corresponding to $i = 1$

parent(i) =i/2: returns index of node's parent

left(i)=2i: returns index of node's left child

right(i)=2i+1: returns index of node's right child



Heap Operations

build_max_heap : produce a max-heap from an unordered array

max_heapify : correct a single violation of the heap property in a subtree at its root

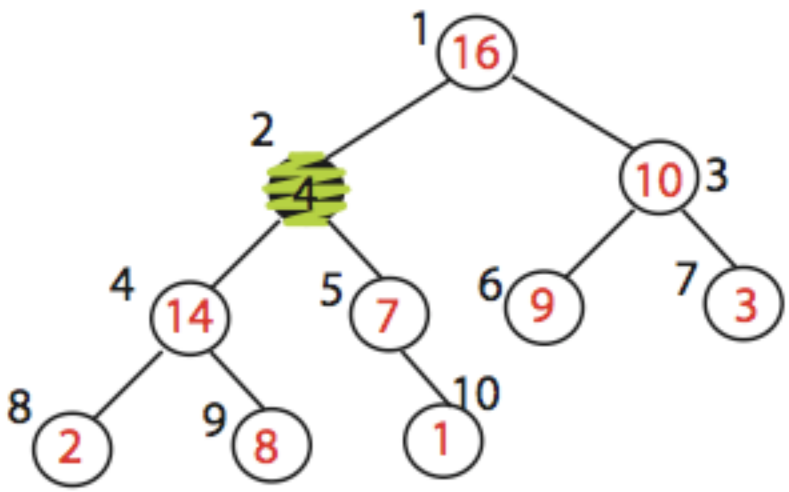
insert, extract_max, heapsort

Max_heapify

- Assume that the trees rooted at left(i) and right(i) are max-heaps

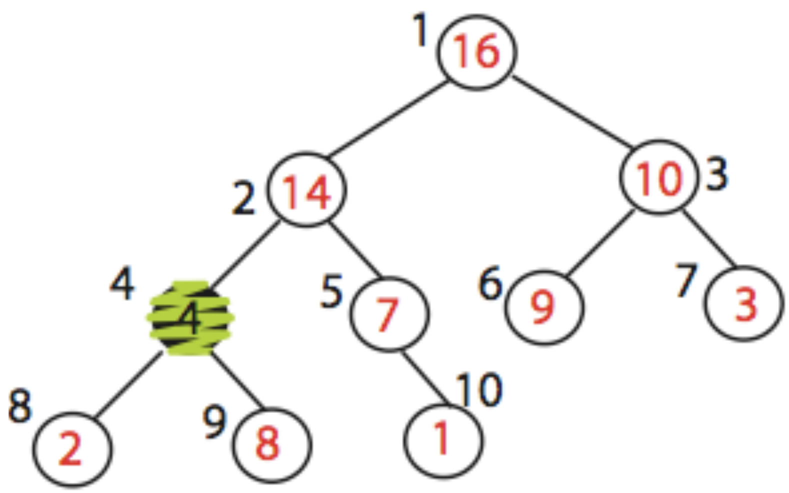
- If element A[i] violates the max-heap property, correct violation by "trickling" element A[i] down the tree, making the subtree rooted at index i a max-heap

Max_heapify (Example)



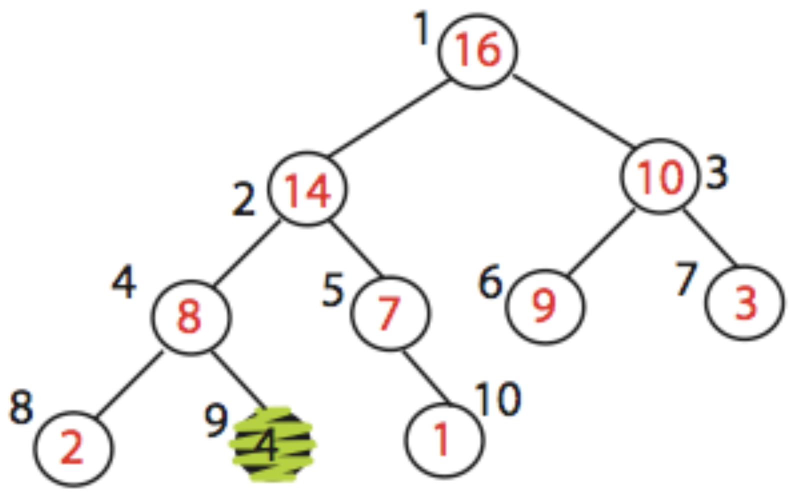
MAX_HEAPIFY (A,2)
heap_size[A] = 10

Max_heapify (Example)



Exchange A[2] with A[4]
Call MAX_HEAPIFY(A,4)
because max_heap property
is violated

Max_heapify (Example)

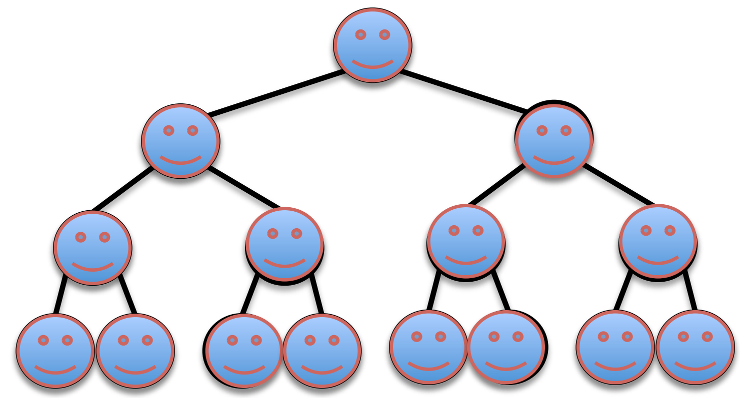


Exchange A[4] with A[9]
No more calls

Time=? O(log n)

Build_Max_Heap(A)

Converts A[1…*n*] to a max heap



Build_Max_Heap(A):
    for i=n/2 downto 1
        do Max_Heapify(A,i)

Time=?   O(n)

T(n)=2T(n/2)+O(log n)   + Master Theorem

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

2. Find maximum element A[1];

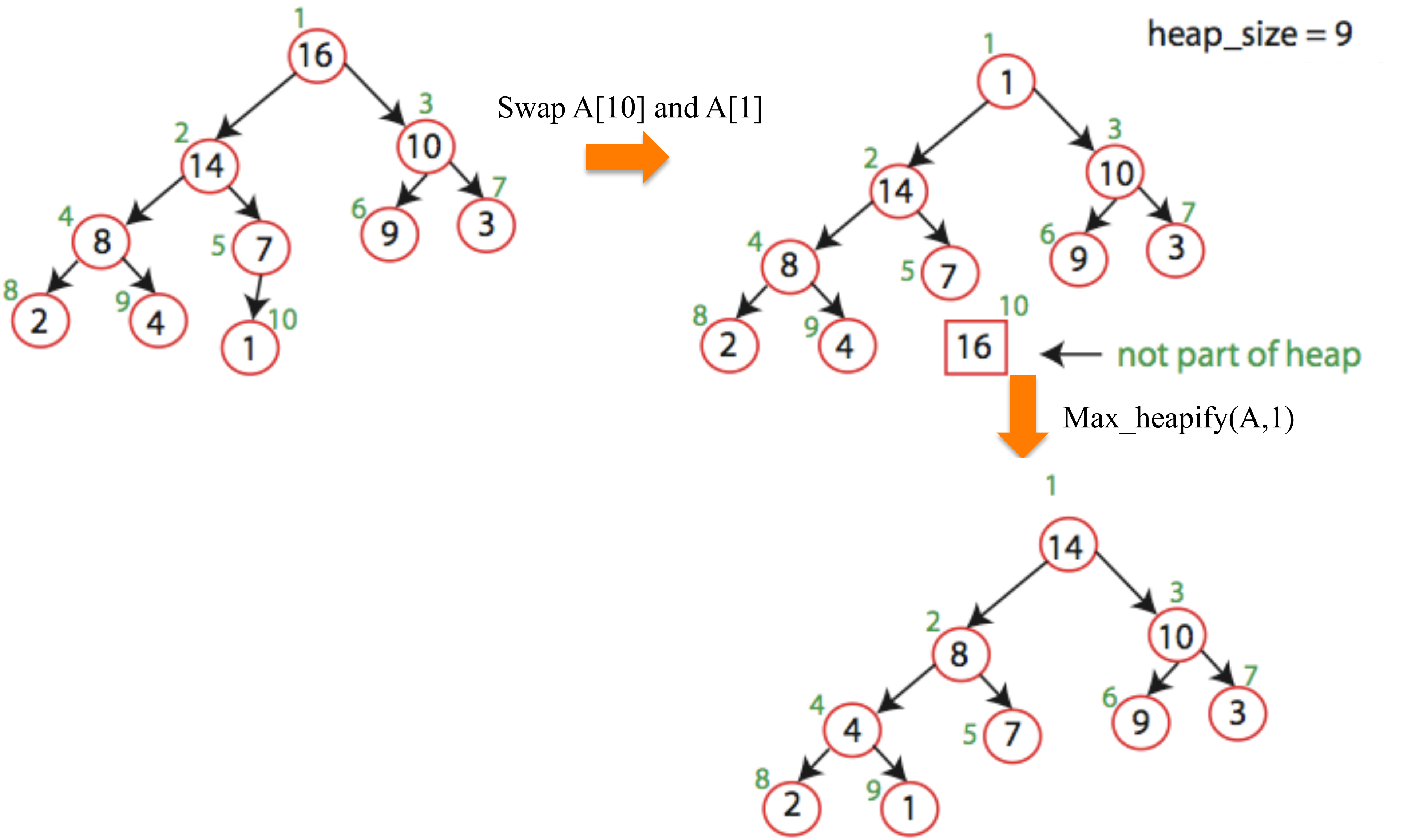
3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

4. Discard node n from heap
   (by decrementing heap-size variable)

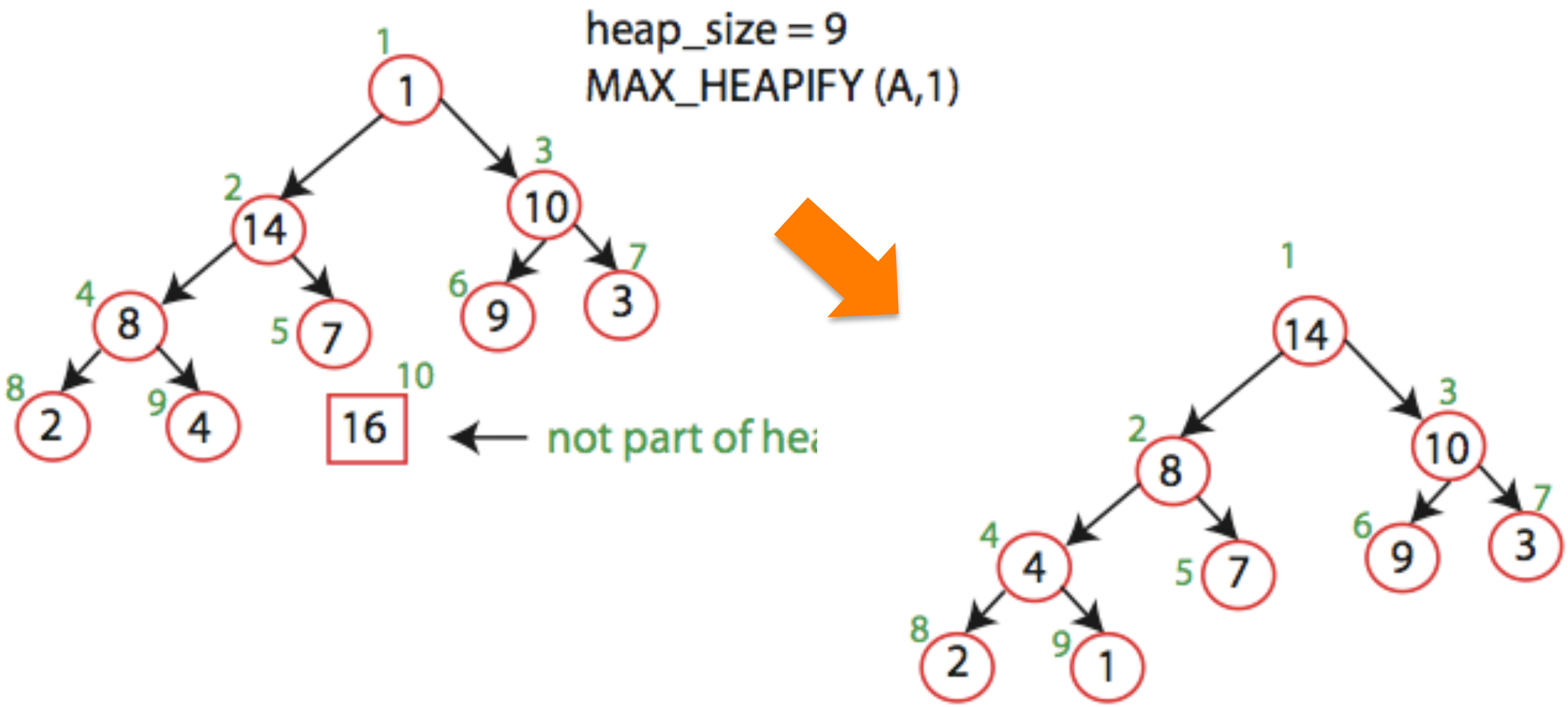
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

6. Go to step 2.

Heap-Sort Demo

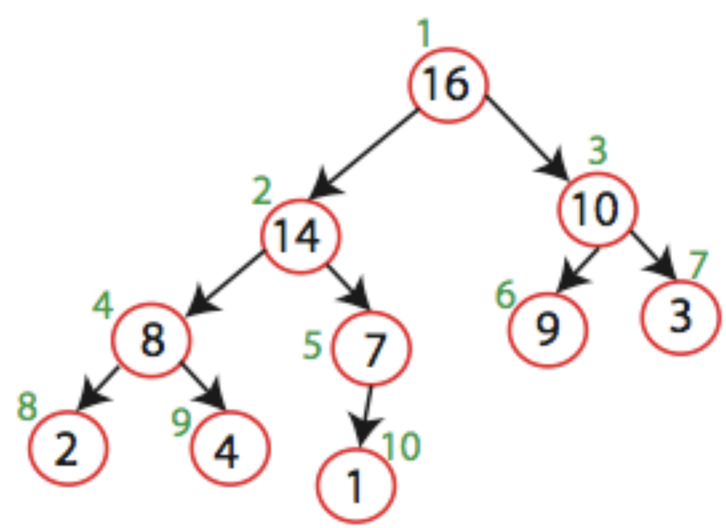


Heap-Sort


heap_size = 9
MAX_HEAPIFY (A,1)
1
2
3
4
5
6
7
8
9
10
1
14
10
8
7
9
3
2
4
16
not part of hea
1
2
3
4
5
6
7
8
9
14
8
10
4
7
9
3
2
1

Heap-Sort

A	4	1	3	2	16	9	10	14	8	7



Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

2. Find maximum element A[1];

3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;
2. Find maximum element A[1];
3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!
4. Discard node n from heap
   (by decrementing heap-size variable)

Heap-Sort

Sorting Strategy:

1. Build Max Heap from unordered array;

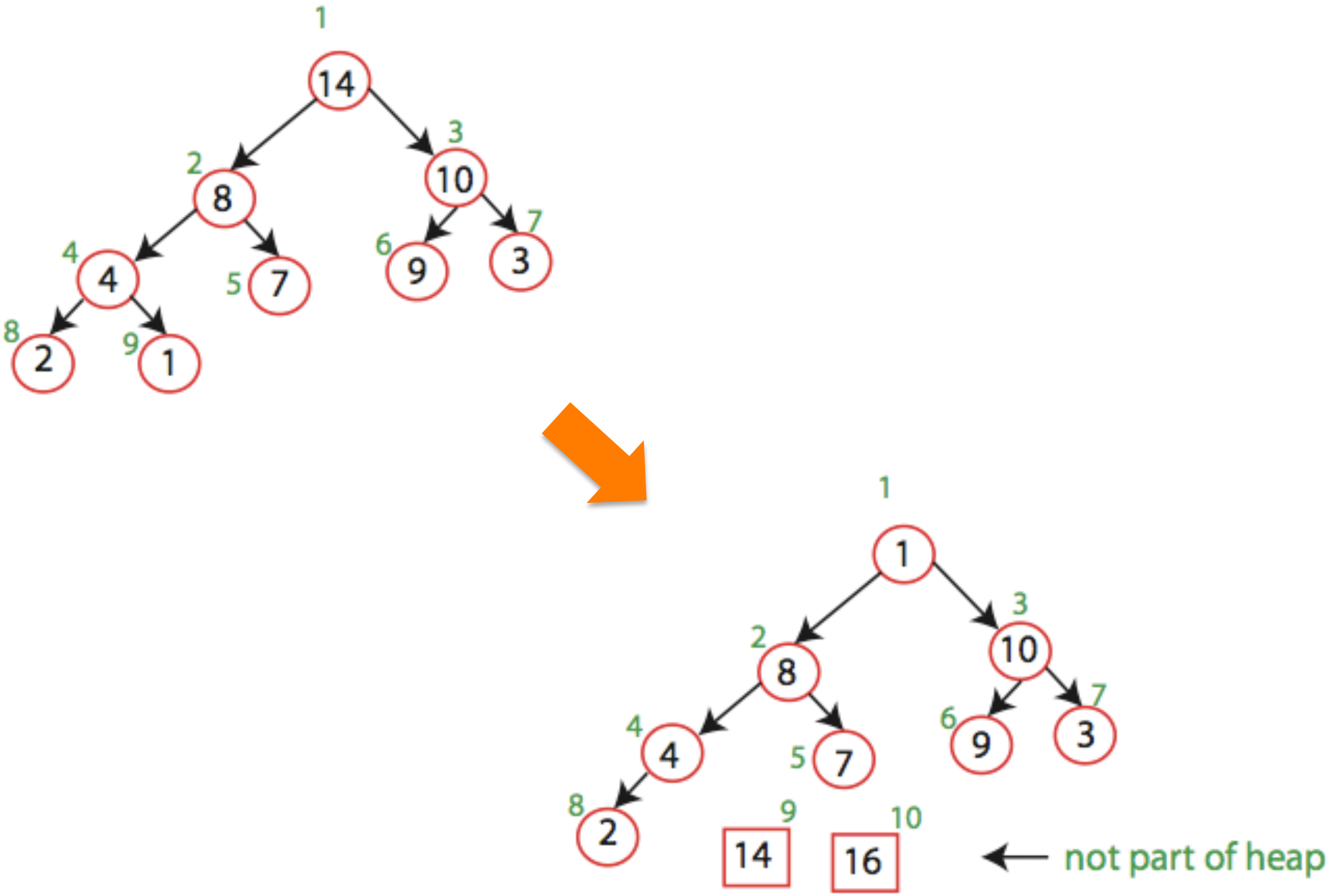
2. Find maximum element A[1];

3. Swap elements A[n] and A[1]:
   now max element is at the end of the array!

4. Discard node n from heap
   (by decrementing heap-size variable)

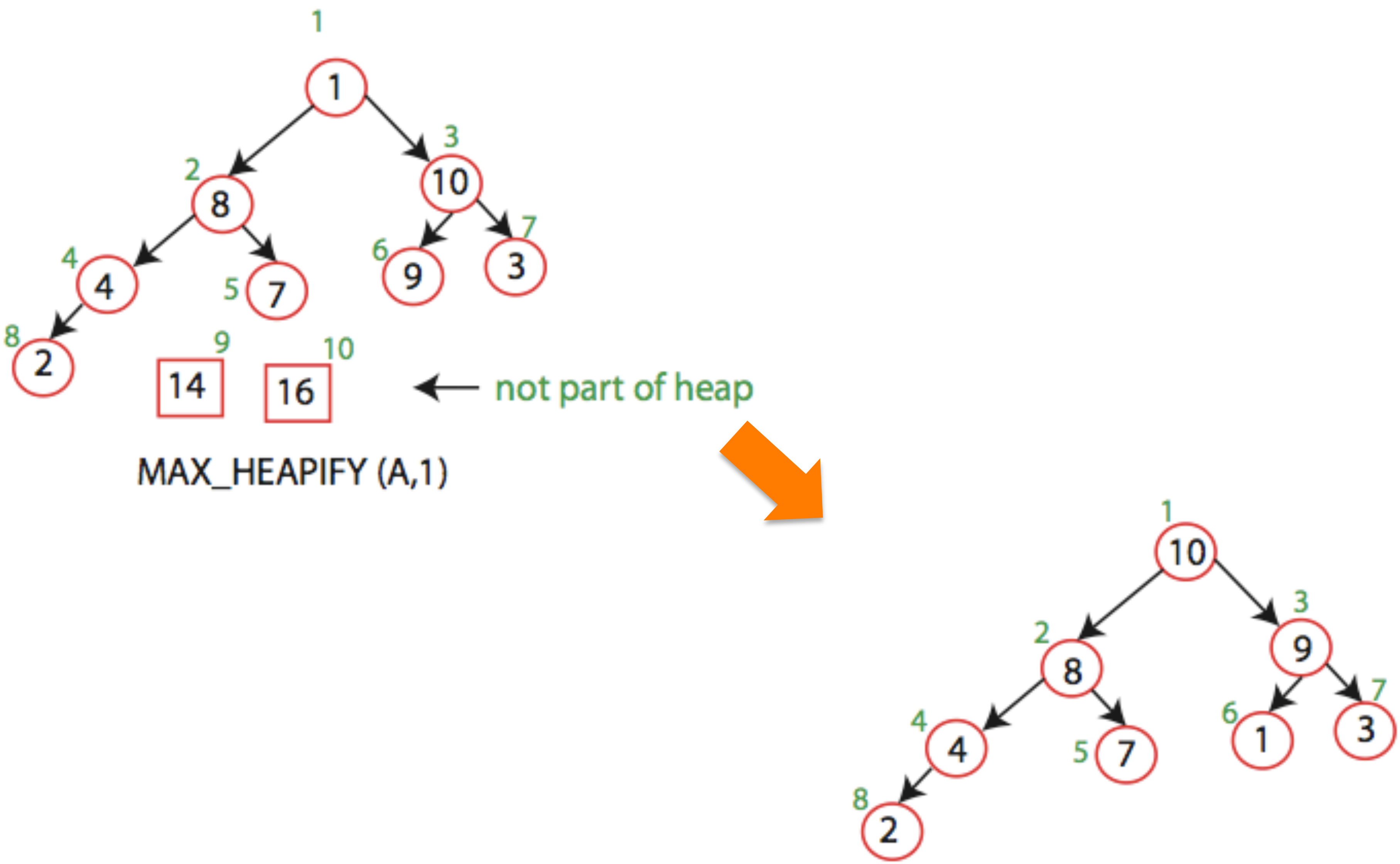
5. New root may violate max heap property, but its children are max heaps. Run max_heapify to fix this.

Heap-Sort



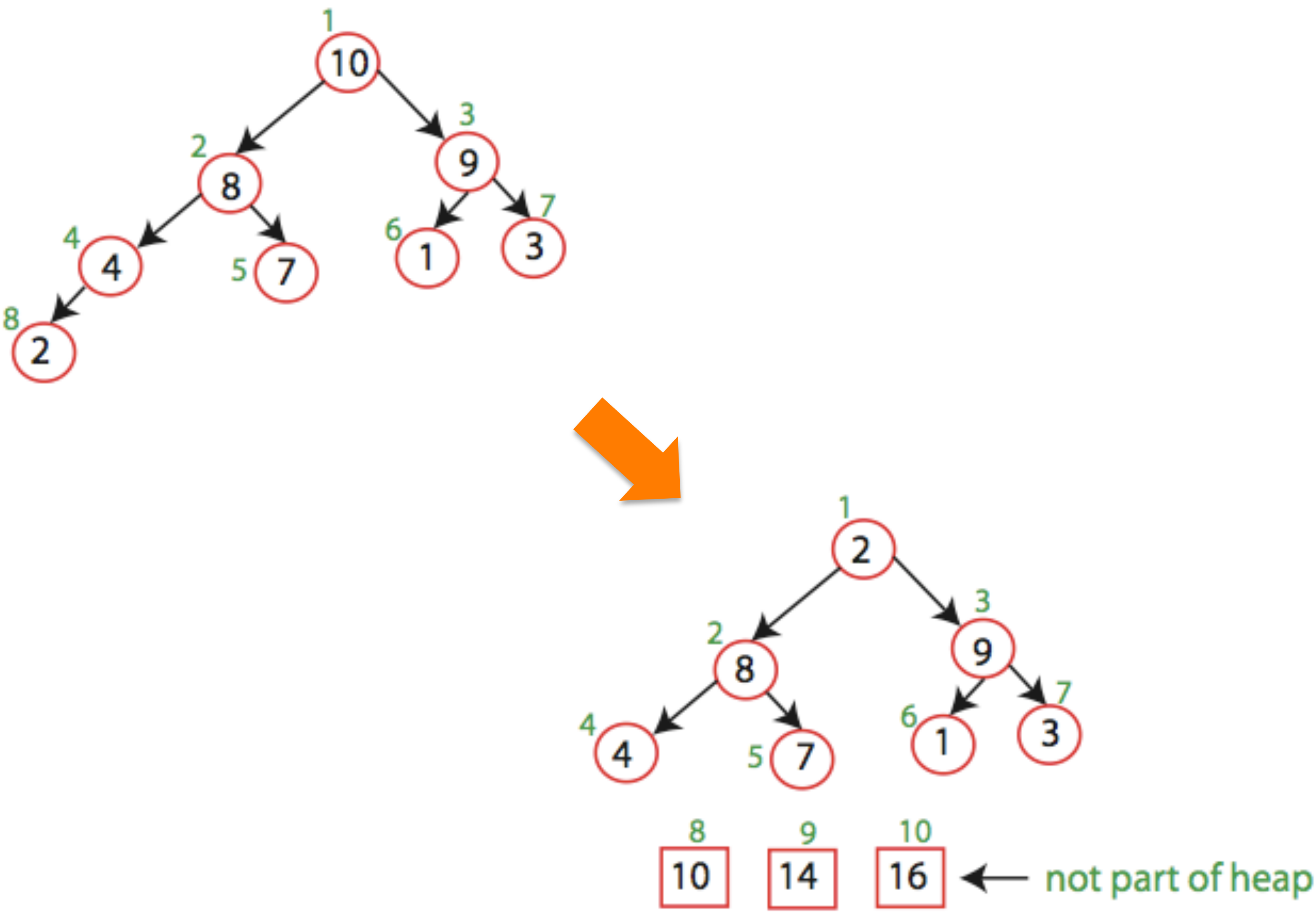
MAX_HEAPIFY (A,1)

Heap-Sort



1
1
2
8
3
10
4
4
5
7
6
9
7
3
8
2
9
14
10
16
not part of heap
MAX_HEAPIFY (A,1)
1
10
2
8
3
9
4
4
5
7
6
1
7
3
8
2

Heap-Sort



Heap-Sort

Running time:

after n iterations the Heap is empty

every iteration involves a swap and a heapify operation; hence it takes $O(\log n)$ time

Overall $O(n \log n)$