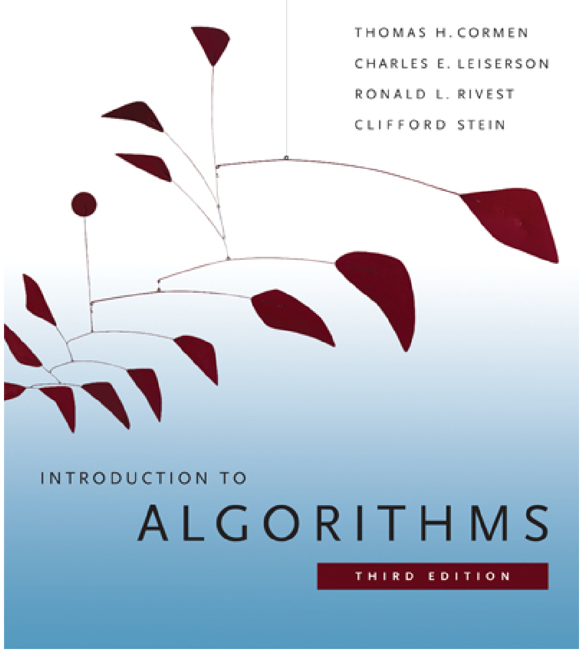
6.006- *Introduction to Algorithms*



Lecture 8

Prof. Piotr Indyk

Menu

- Sorting!
  - Insertion Sort
  - Merge Sort
- Recurrences
  - Master theorem

The problem of sorting

Input: array A[1…n] of numbers.

Output: permutation B[1…n] of A such that B[1] £ B[2] £ ⋯ £ B[n] .

e.g. A = [7, 2, 5, 5, 9.6] → B = [2, 5, 5, 7, 9.6]

How can we do it efficiently ?

Insertion sort

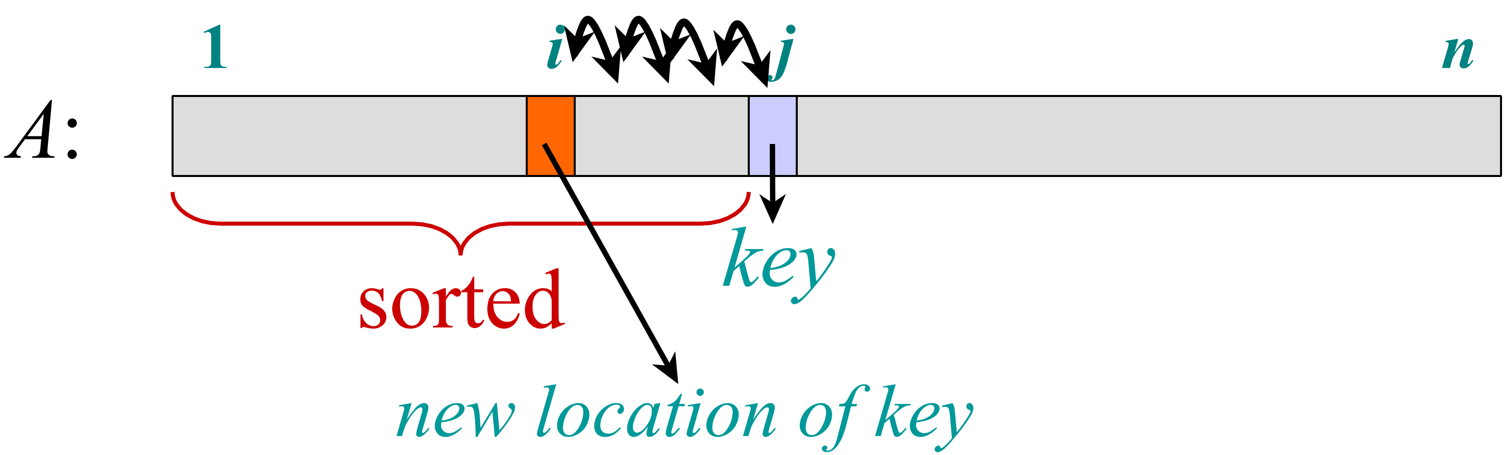
INSERTION-SORT (A, n) ⊳ $A[1 \mathinner{..} n]$

for $j \leftarrow 2$ **to** n

insert key $A[j]$ **into the (already sorted) sub-array** $A[1 \mathinner{..} j\text{-}1]$.

by pairwise key-swaps down to its right position

Illustration of iteration $\boldsymbol{j}$



Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8 2 4 9 3 6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

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8	2	4	9	3	6
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8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

Example of insertion sort

8	2	4	9	3	6
2	8	4	9	3	6
2	4	8	9	3	6
2	4	8	9	3	6
2	3	4	8	9	6

Example of insertion sort

8 2 4 9 3 6

2 8 4 9 3 6

2 4 8 9 3 6

2 4 8 9 3 6

2 3 4 8 9 6

2 3 4 6 8 9 *done*

Running time? $\Theta(n^2)$

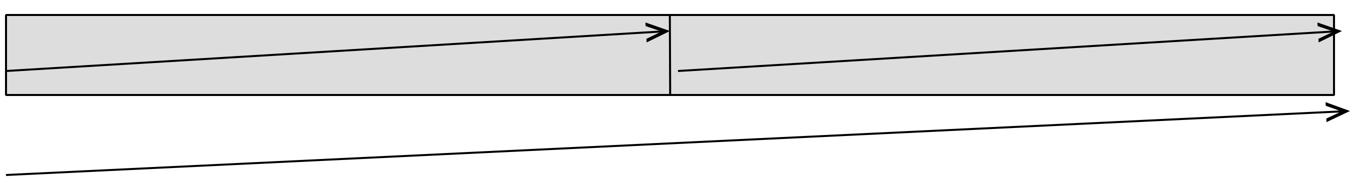
e.g. when input is $A = [n, n-1, n-2, \ldots, 2, 1]$

Meet Merge Sort

divide and conquer

MERGE-SORT $A[1 \,.\,.\, n]$

1. If $n = 1$, done (nothing to sort).
2. Otherwise, recursively sort $A[\, 1 \,.\,.\, n/2 \,]$ and $A[\, n/2+1 \,.\,.\, n \,]$.
3. "***Merge***" the two sorted sub-arrays.



Key subroutine: **MERGE**

Merging two sorted arrays

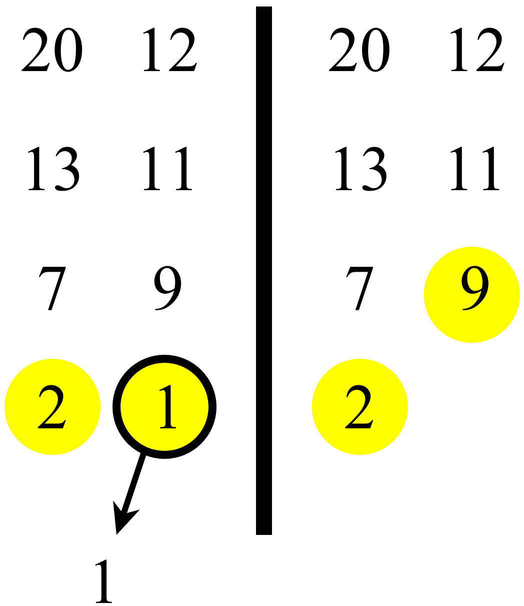
20	12
13	11
7	9
2	1

Merging two sorted arrays

20	12
13	11
7	9
2	1

1

Merging two sorted arrays



Merging two sorted arrays

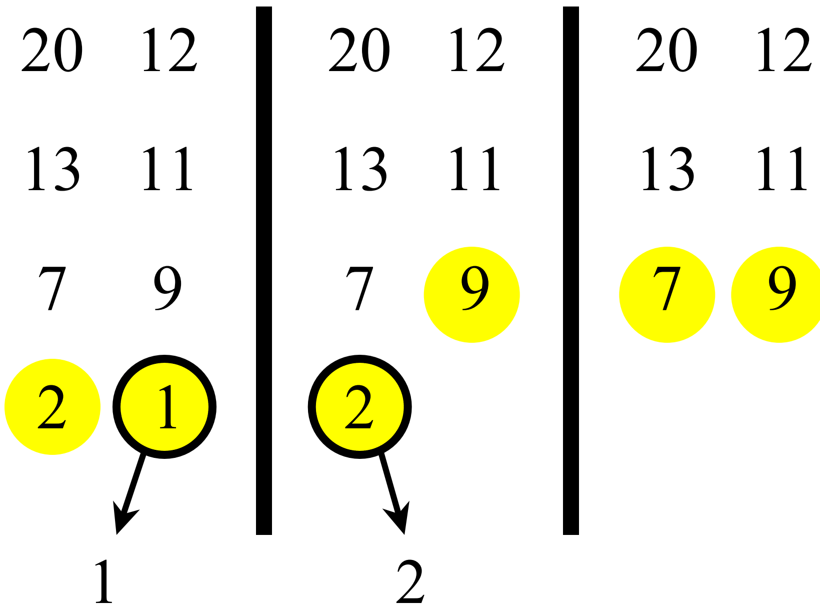
20	12
13	11
7	9
2	1

1

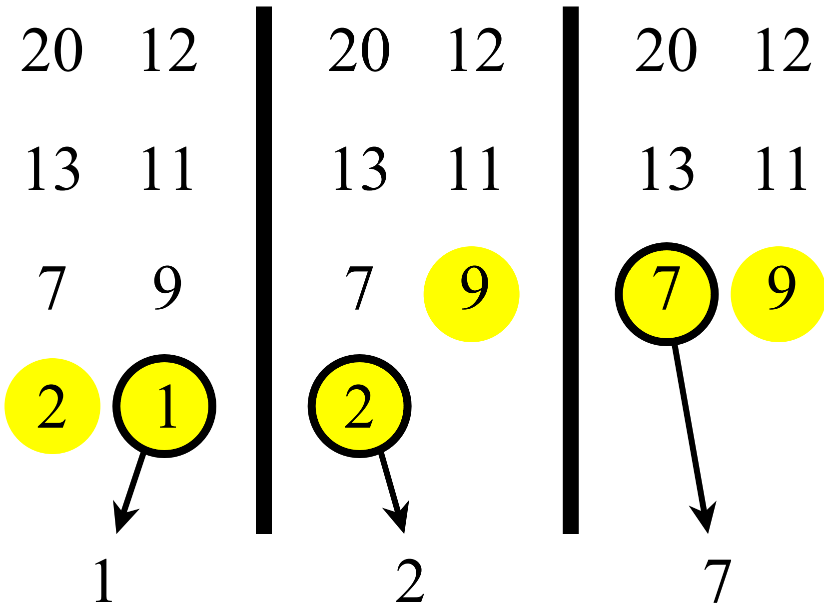
20	12
13	11
7	9
2	

2

Merging two sorted arrays



Merging two sorted arrays



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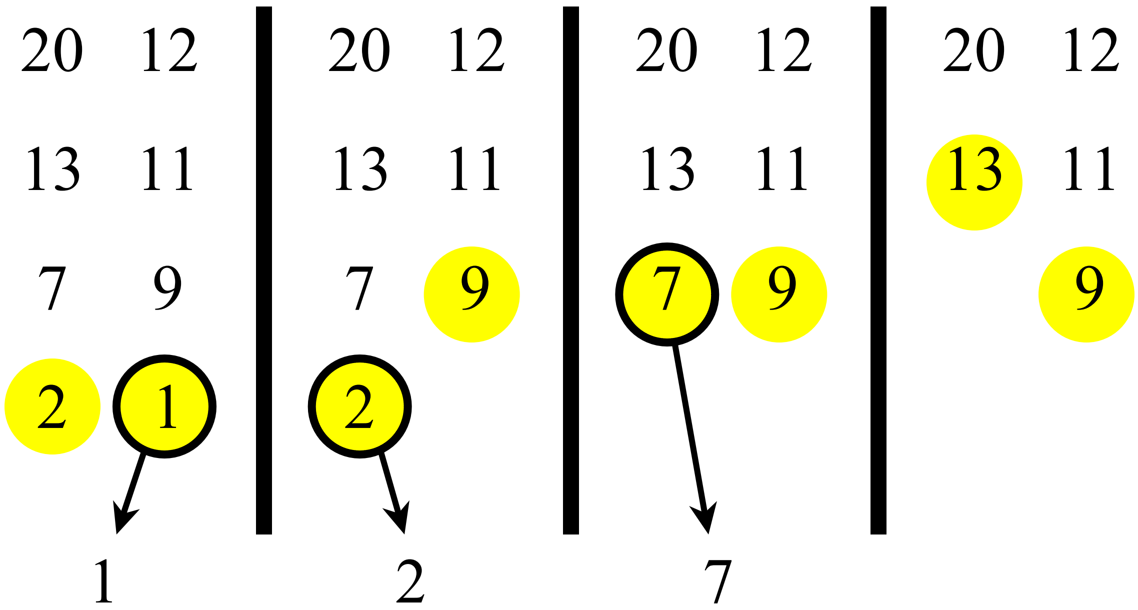

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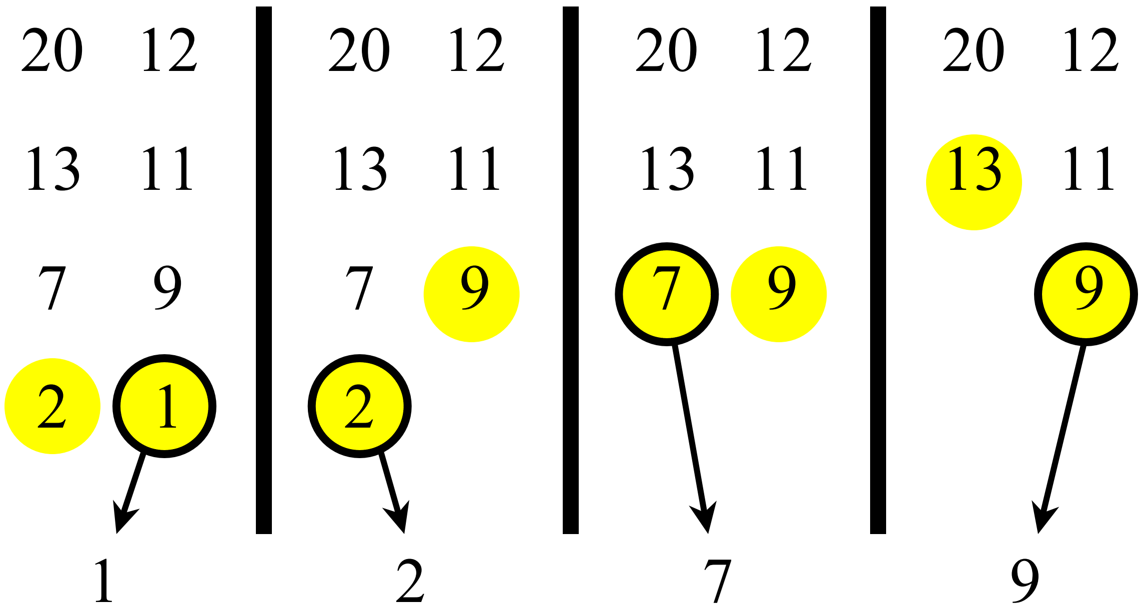
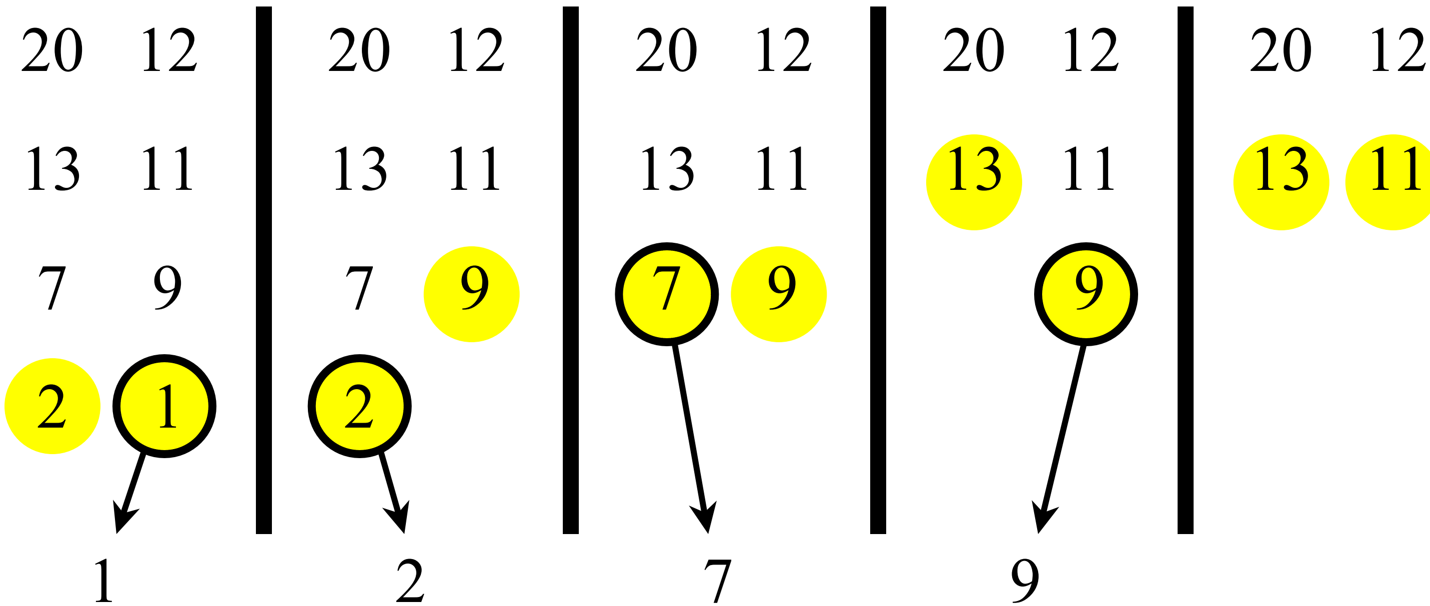
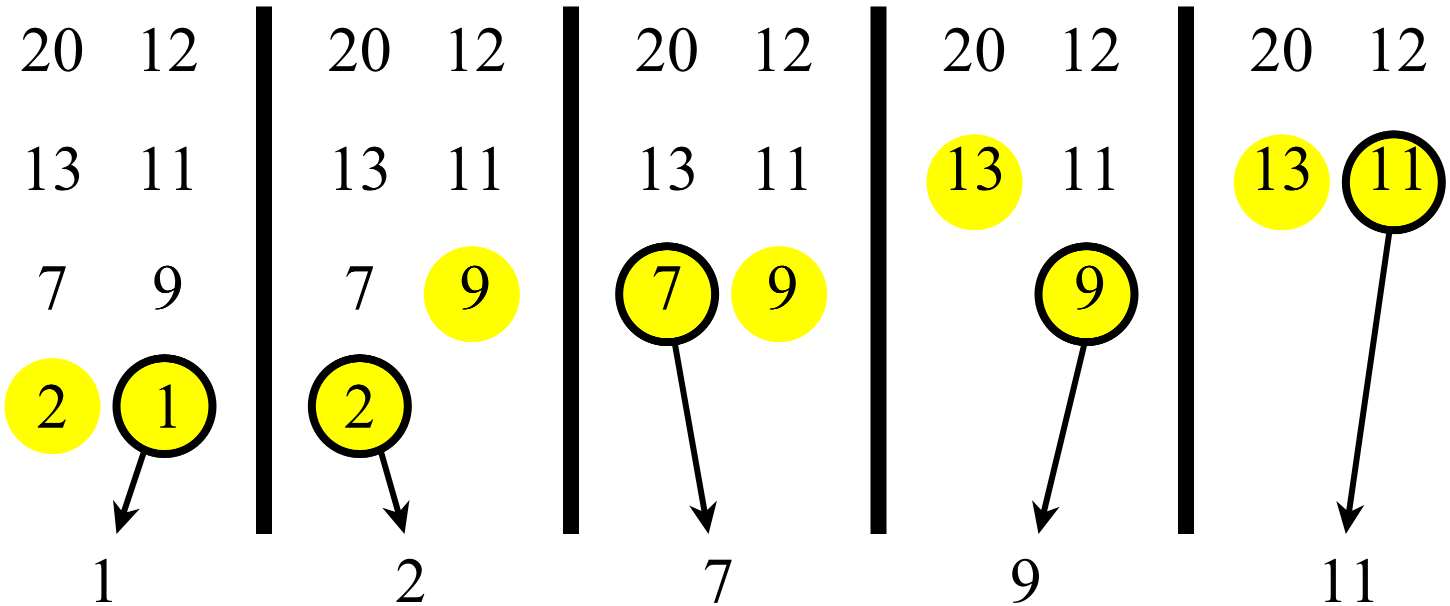


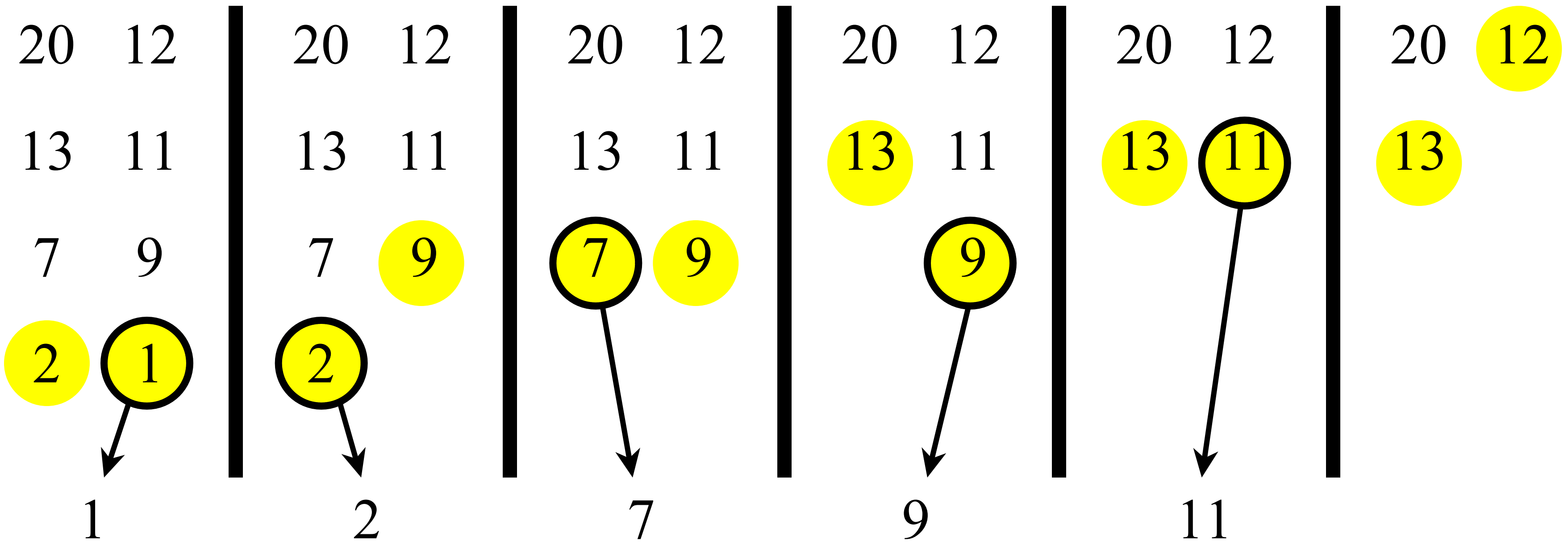
Merging two sorted arrays



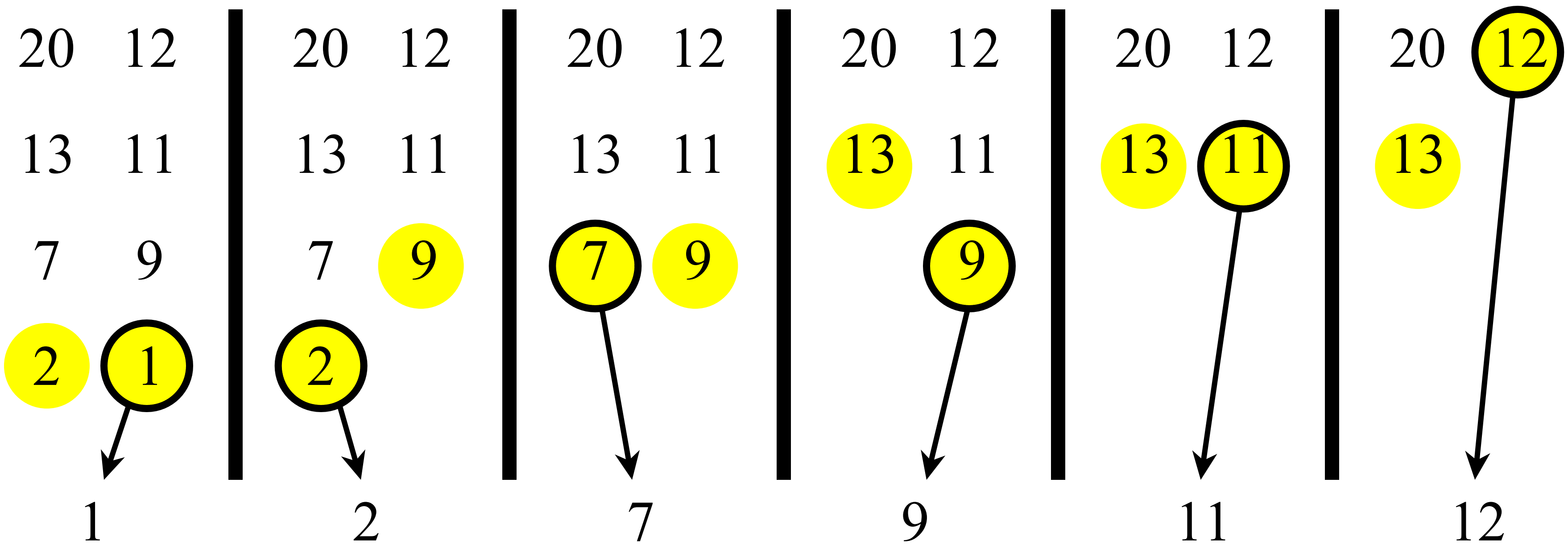
Merging two sorted arrays



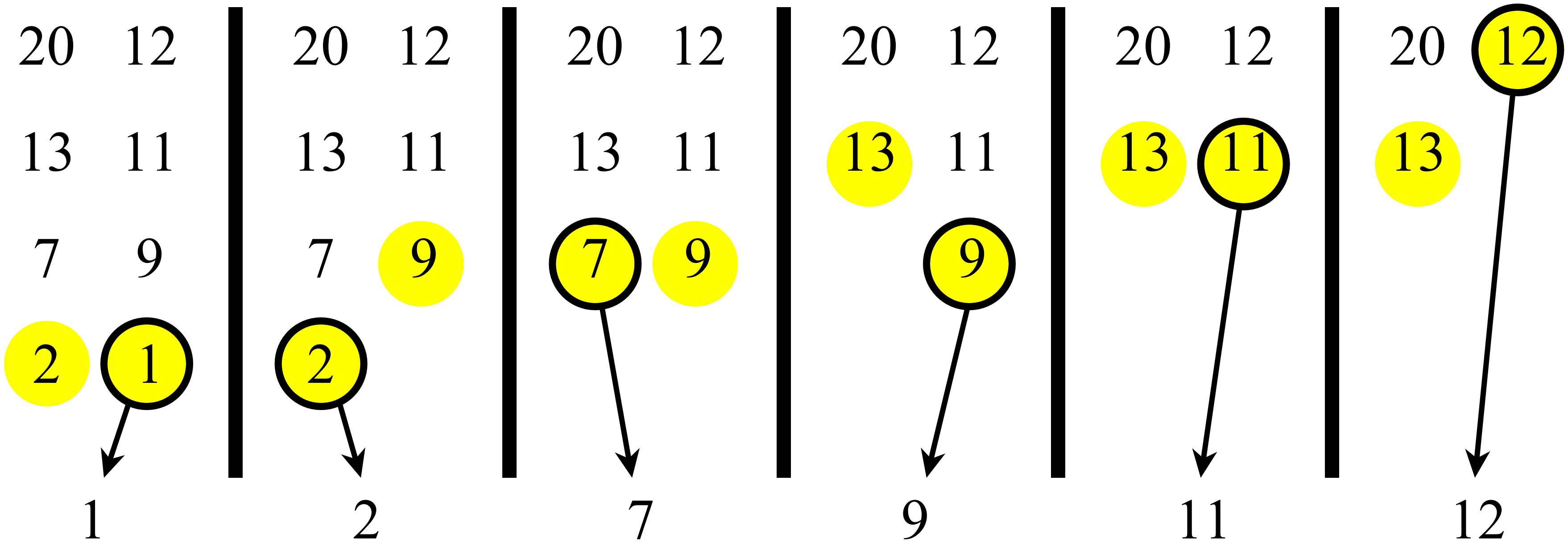
Merging two sorted arrays



Merging two sorted arrays



Merging two sorted arrays



Time = $\Theta(n)$ to merge a total of n elements (linear time).

Analyzing merge sort

MERGE-SORT $A[1 \,.\,.\, n]$	$T(n)$
1. If $n = 1$, done.	$\mathrm{Q}(1)$
2. Recursively sort $A[\,1 \,.\,.\, \lceil n/2 \rceil\,]$ and $A[\,\lceil n/2 \rceil + 1 \,.\,.\, n\,]$.	$2T(n/2)$
3. ***"Merge"*** the two sorted lists	$\mathrm{Q}(n)$

$$T(n) = \begin{cases} \mathrm{Q}(1) \text{ if } n = 1; \\ 2T(n/2) + \mathrm{Q}(n) \text{ if } n > 1. \end{cases}$$

$$T(n) = ?$$

Recurrence solving

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

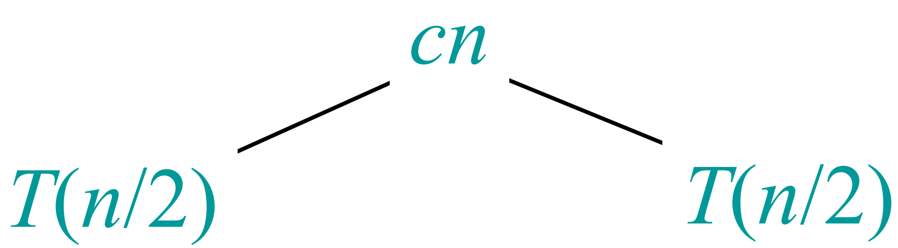
Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.

$$T(n)$$

Recursion tree

Solve $T(n) = 2T(n/2) + cn$, where $c > 0$ is constant.



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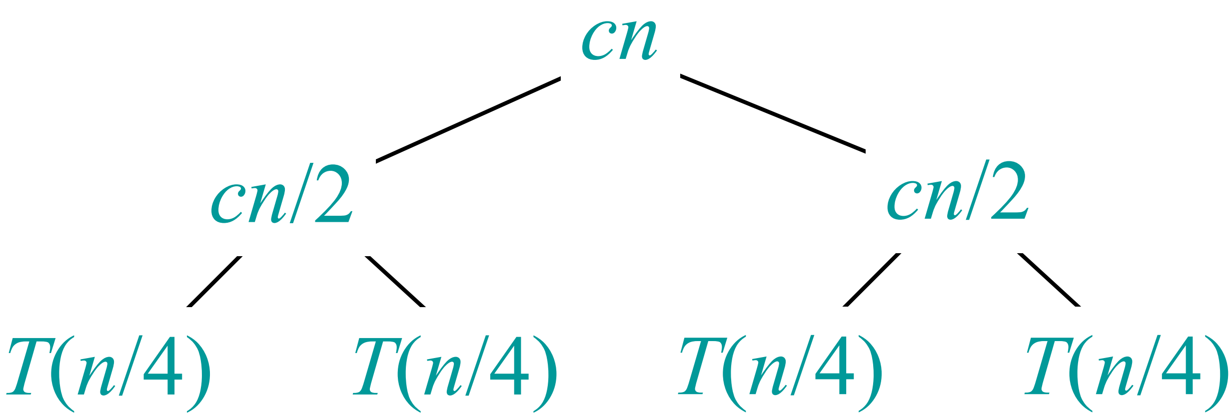

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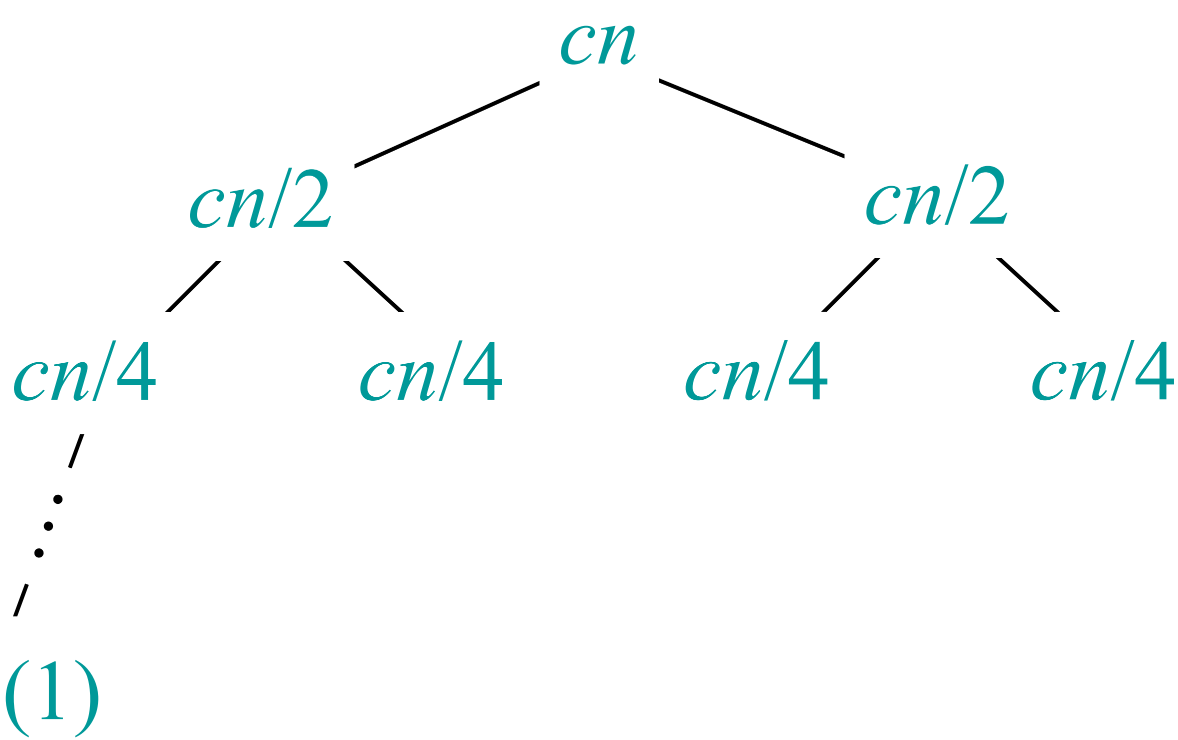

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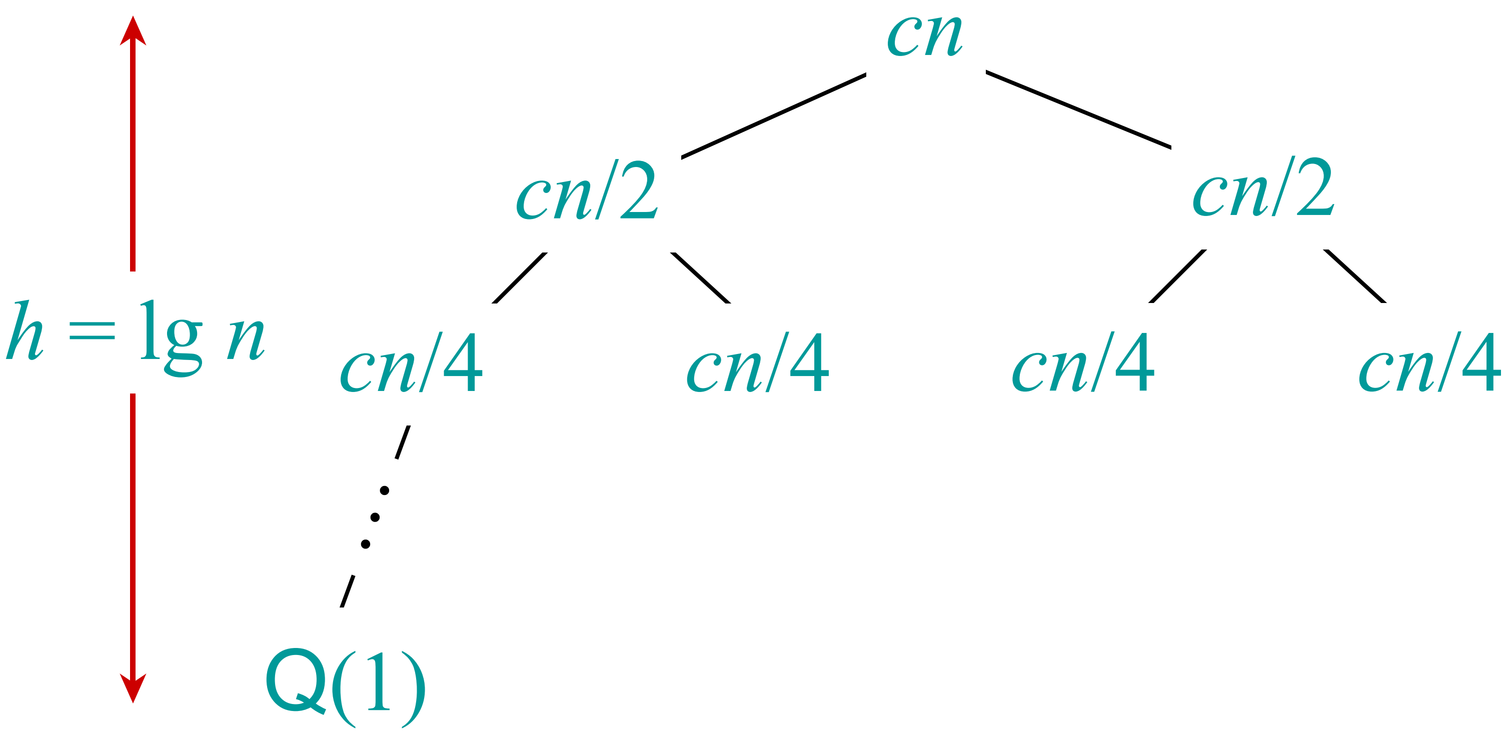

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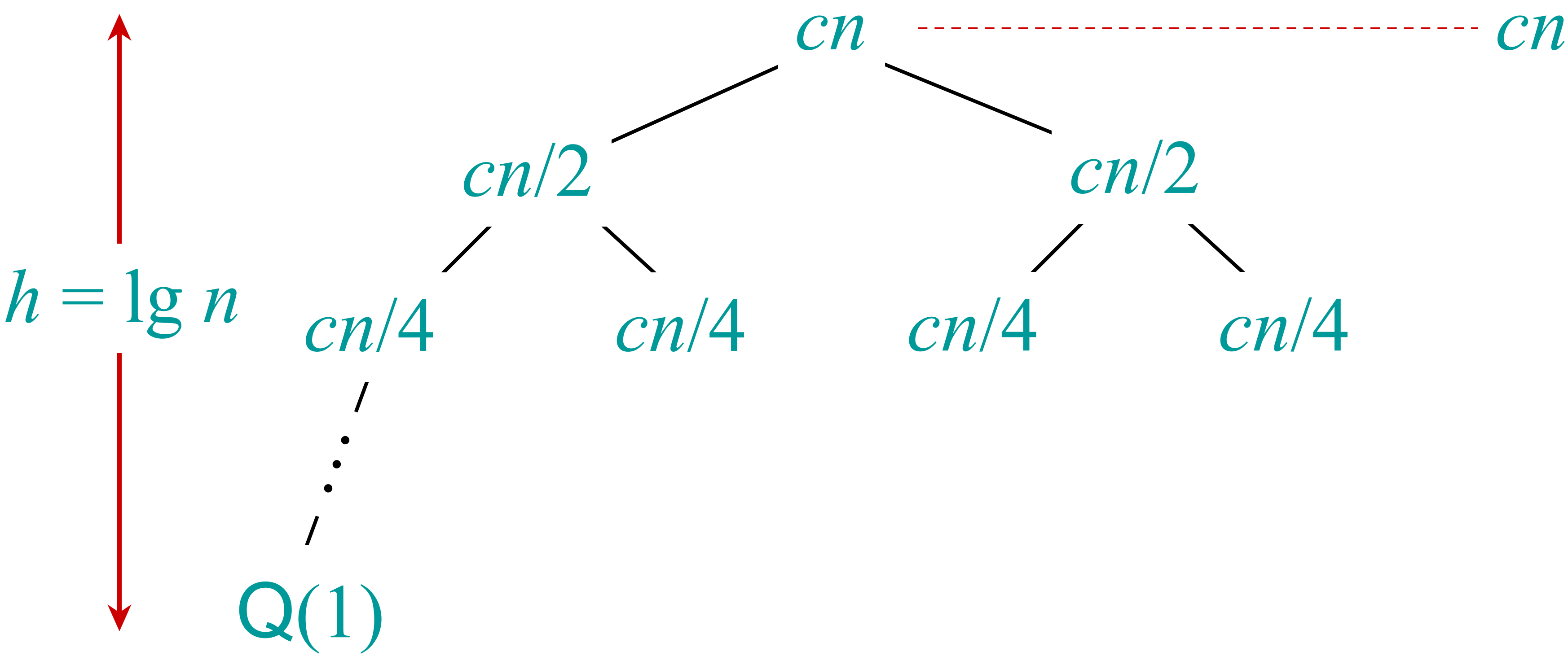

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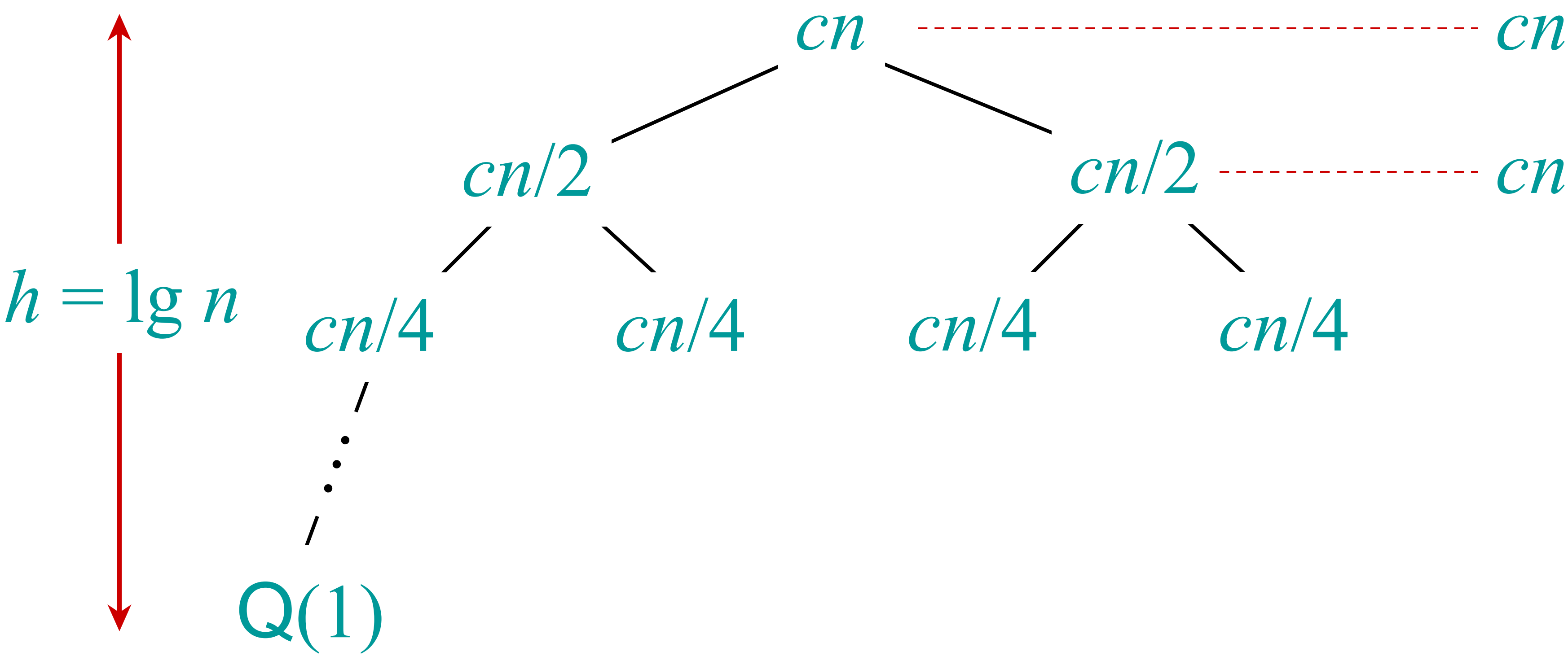

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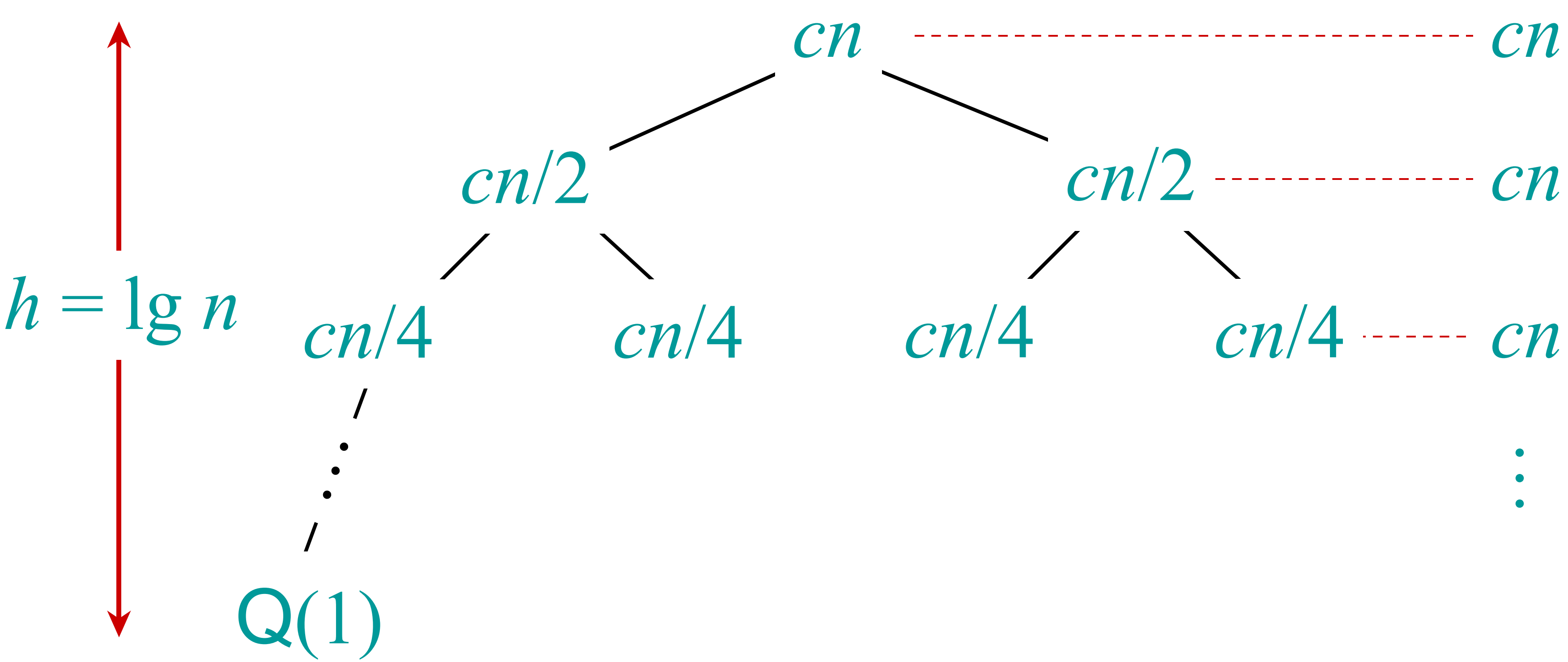

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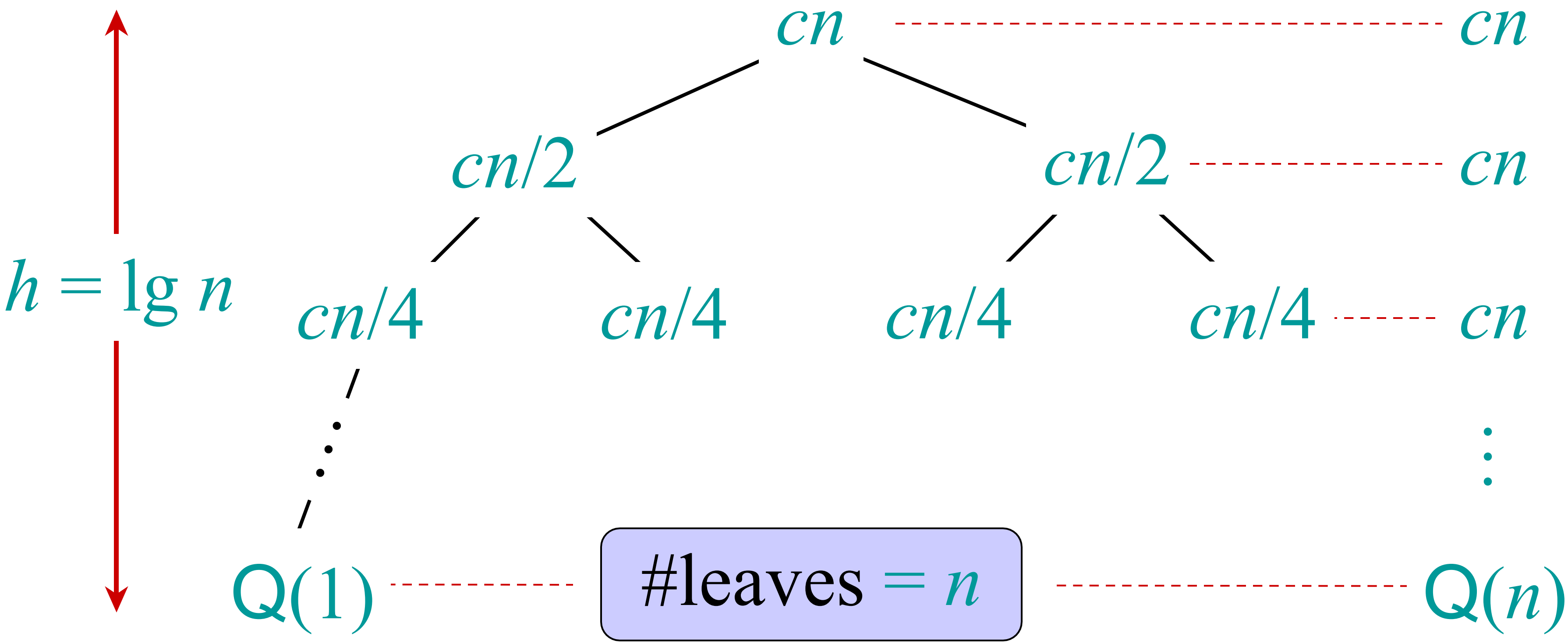

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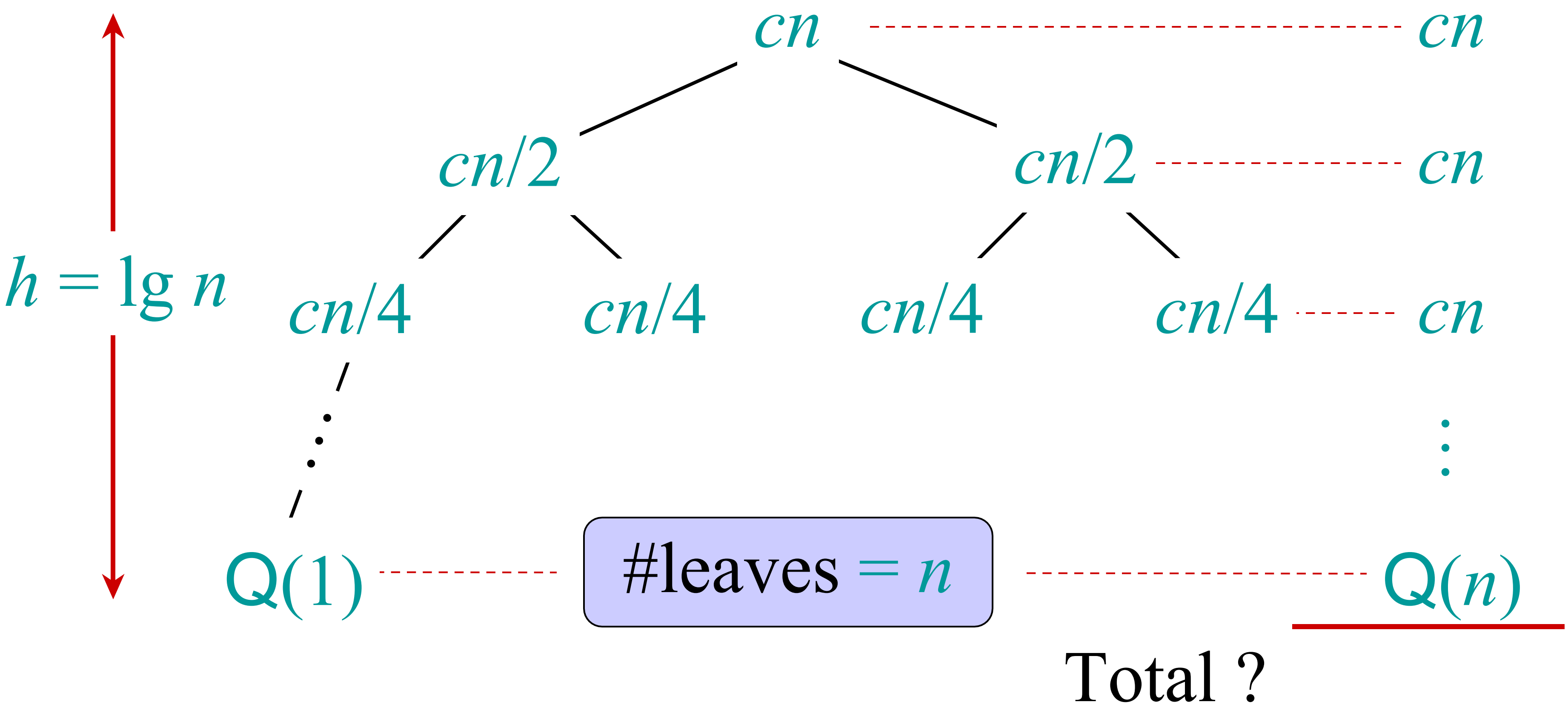

Recursion tree

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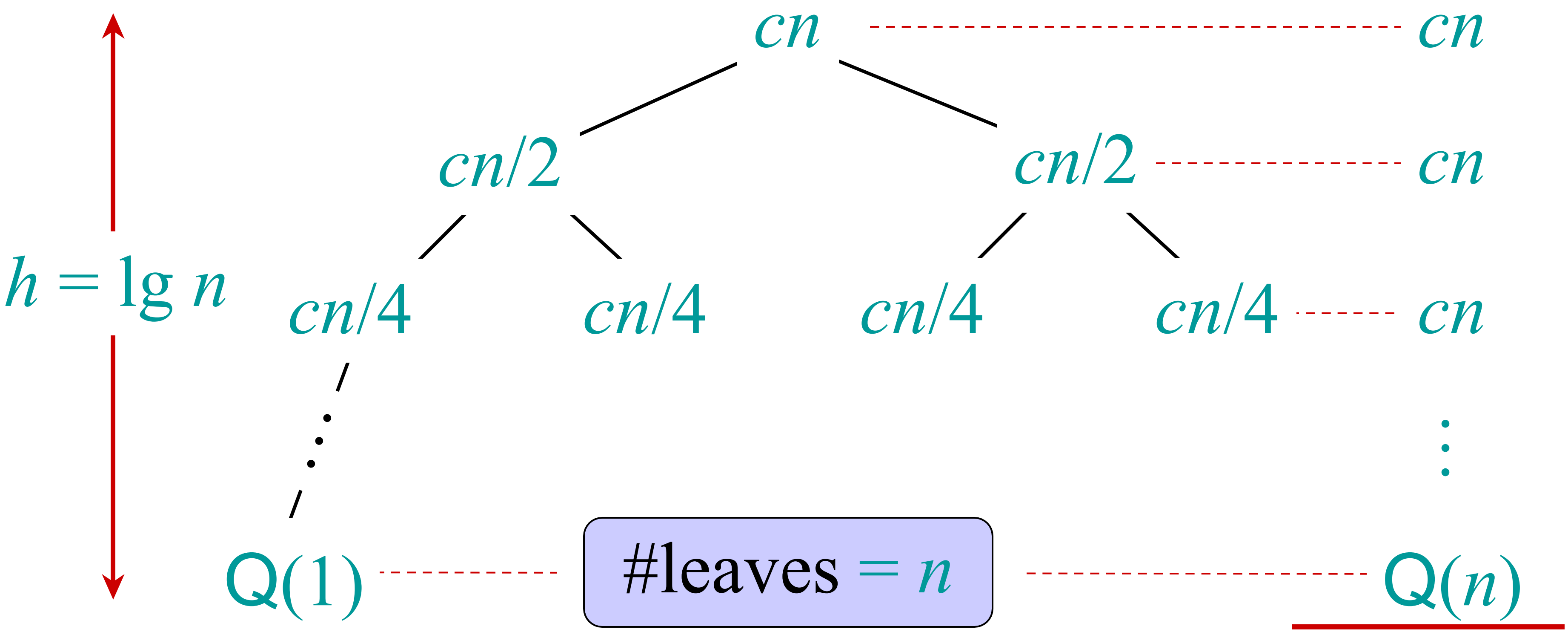


Total = Q(n lg n)

The master method

"One theorem to rule them all" (sort of)

The master method applies to recurrences of the form

$$T(n) = a\,T(n/b) + f(n),$$

#subproblems

size of each subproblem

time to split into subproblems and combine results

where $a \geq 1$, $b > 1$, and f is positive.

e.g. Mergesort: a=2, b=2, $f(n)$=O(n)

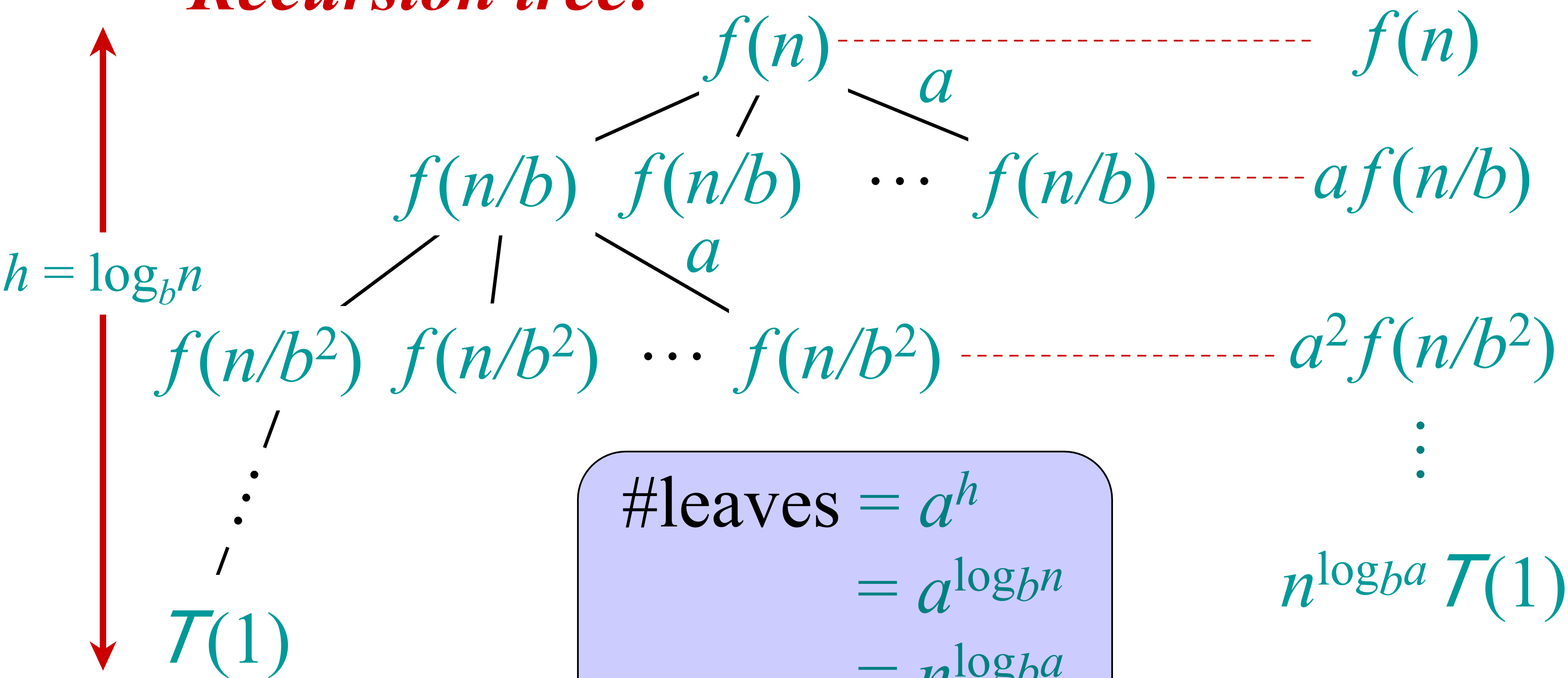
e.g.2 Binary Search: a=1, b=2, $f(n)$=O(1)

Basic Idea: Compare $f(n)$ with $n^{\log_b a}$.

Idea of master theorem

$T(n) = a\,T(n/b) + f(n)$

Recursion tree:



Idea of master theorem

$$T(n) = a\,T(n/b) + f(n)$$

Recursion tree:

$$f(n) \quad \cdots\cdots \quad f(n)$$

$$\overbrace{f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b)}^{a} \quad \cdots\cdots \quad a\,f(n/b)$$

$$\overbrace{f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2)}^{a} \quad \cdots\cdots \quad a^2 f(n/b^2)$$

$$\vdots$$

$$h = \log_b n$$

$$T(1) \qquad\qquad n^{\log_b a}\,T(1)$$

CASE 1: The weight increases geometrically from the root to the leaves. The leaves hold a constant fraction of the total weight.

$$\Theta(n^{\log_b a})$$

Idea of master theorem

$$T(n) = a\,T(n/b) + f(n)$$

Recursion tree:

$f(n)$ ----- $f(n)$

a

$f(n/b)\quad f(n/b)\quad \cdots \quad f(n/b)$ ----- $a\,f(n/b)$

a

$h = \log_b n$

$f(n/b^2)\quad f(n/b^2)\quad \cdots \quad f(n/b^2)$ ----- $a^2 f(n/b^2)$

$\vdots$

$T(1)$

$n^{\log_b a}\,T(1)$

CASE 2: ($k = 0$) The weight is approximately the same on each of the $\log_b n$ levels.

$$\Theta(n^{\log_b a} \lg n)$$

Idea of master theorem

$$T(n) = a\,T(n/b) + f(n)$$

Recursion tree:

$f(n)$ — a — $f(n)$

$f(n/b) \quad f(n/b) \quad \cdots \quad f(n/b)$ — $a\,f(n/b)$

a

$h = \log_b n$

$f(n/b^2) \quad f(n/b^2) \quad \cdots \quad f(n/b^2)$ — $a^2 f(n/b^2)$

$\vdots$

$T(1)$ — $n^{\log_b a}\,T(1)$

CASE 3: The weight decreases geometrically from the root to the leaves. The root holds a constant fraction of the total weight.

$\Theta(f(n))$

Three common cases

Compare $f(n)$ with $n^{\log_b a}$:

1. $f(n) = \Theta(n^{\log_b a - \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially slower than $n^{\log_b a}$ (by an n^{ε} factor).

   cost of level $i = a^i f(n/b^i) = \Theta(n^{\log_b a - \varepsilon} \cdot b^{i\varepsilon})$

   so geometric increase of cost as we go deeper in the tree

   hence, leaf level cost dominates!

Solution: $T(n) = \Theta(n^{\log_b a})$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

2. $f(n) = \Theta(n^{\log_b a} \log^k n)$ for some constant $k \geq 0$.
   - $f(n)$ and $n^{\log_b a}$ grow at similar rates.

(cost of level i) = $a^i f(n/b^i) = \Theta(n^{\log_b a} \cdot \log^k (n/b^i))$
so all levels have about the same cost

Solution: $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.

Three common cases (cont.)

Compare $f(n)$ with $n^{\log_b a}$:

3. $f(n) = \Theta(n^{\log_b a + \varepsilon})$ for some constant $\varepsilon > 0$.
   - $f(n)$ grows polynomially faster than $n^{\log_b a}$ (by an n^{ε} factor).

   (cost of level i) = $a^i f(n/b^i) = \Theta(n^{\log_b a+\varepsilon} \cdot b^{-i\,\varepsilon})$
   so geometric decrease of cost as we go deeper in the tree
   hence, root cost dominates!

   Solution: $T(n) = \Theta(f(n))$.

Examples

Ex. $T(n) = 2T(n/2) + 1$

$a = 2, b = 2 \Rightarrow n^{\log_b a} = n; f(n) = 1.$

Case 1: $f(n) = O(n^{1-\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n).$

Ex. $T(n) = 2T(n/2) + n$

$a = 2, b = 2 \Rightarrow n^{\log_b a} = n; f(n) = n.$

Case 2: $f(n) = \Theta(n \lg^0 n)$, that is, $k = 0$.

$\therefore\ T(n) = \Theta(n \lg n).$

Examples

Ex. $T(n) = 4T(n/2) + n^3$

$a = 4$, $b = 2 \Rightarrow n^{\log_b a} = n^2$; $f(n) = n^3$.

CASE 3: $f(n) = \Omega(n^{2+\varepsilon})$ for $\varepsilon = 1$.

$\therefore\ T(n) = \Theta(n^3)$.