Quiz 2

- Do not open this quiz booklet until directed to do so. Read all the instructions on this page.
- When the quiz begins, write your name on every page of this quiz booklet.
- You have 120 minutes to earn **120** points. Do not spend too much time on any one problem. Read them all through first, and attack them in the order that allows you to make the most progress.
- This quiz booklet contains 19 pages, including this one. Two extra sheets of scratch paper are attached. Please detach them before turning in your quiz at the end of the exam period.
- This quiz is closed book. You may use **one** handwritten, $8\frac{1}{2}'' \times 11''$ or A4 crib sheets (both sides). No calculators or programmable devices are permitted. No cell phones or other communications devices are permitted.
- Write your solutions in the space provided. If you need more space, write on the back of the sheet containing the problem. Pages may be separated for grading.
- Do not waste time and paper rederiving facts that we have studied. It is sufficient to cite known results.
- Show your work, as partial credit will be given. You will be graded not only on the correctness of your answer, but also on the clarity with which you express it. Be neat.
- Good luck!

Problem	Parts	Points	Grade	Grader		Problem	Parts	Points	Grade	Grader
1	8	24				6	–	10		
2	6	24				7	–	10		
3	4	12				8	–	10		
4	–	10				9	–	10		
5	–	10								
						Total		120		

Name: _____

Athena username: _____

Recitation:	Nick	Nick	Tianren	David	Joe	Joe	Michael
	WF10	**WF11**	**WF12**	**WF1**	**WF2**	**WF3a**	**WF3b**

Problem 1. True or false [24 points] (8 parts)

For each of the following questions, circle either T (True) or F (False). **Explain your choice.** (Your explanation is worth more than your choice of true or false.)

(a) T F Instead of using counting sort to sort digits in the radix sort algorithm, we can use any valid sorting algorithm and radix sort will still sort correctly.

(b) T F The depth of a breadth-first search tree on an undirected graph $G = (V, E)$ from an arbitrary vertex $v \in V$ is the diameter of the graph G. (The ***diameter*** d of a graph is the smallest d such that every pair of vertices s and t have $\delta(s, t) \leq d$.)

(c) T F Every directed acyclic graph has exactly one topological ordering. has valid topological orderings $[a, b, c]$ or $[a, c, b]$. As another example, $G = (V, E) = (\{a, b\}, \{\})$ has valid topological orderings $[a, b]$ or $[b, c]$.

(d) T F Given a graph $G = (V, E)$ with positive edge weights, the Bellman-Ford algorithm and Dijkstra's algorithm can produce different shortest-path trees despite always producing the same shortest-path weights.

(e) T F Dijkstra's algorithm may not terminate if the graph contains negative-weight edges.

(f) T F Consider a weighted directed graph $G = (V, E, w)$ and let X be a shortest s-t path for $s, t \in V$. If we double the weight of every edge in the graph, setting $w'(e) = 2w(e)$ for each $e \in E$, then X will still be a shortest s-t path in (V, E, w').

(g) T F If a depth-first search on a directed graph $G = (V, E)$ produces exactly one back edge, then it is possible to choose an edge $e \in E$ such that the graph $G' = (V, E - \{e\})$ is acyclic.

(h) T F If a directed graph G is cyclic but can be made acyclic by removing one edge, then a depth-first search in G will encounter exactly one back edge.

Problem 2. Short answer [24 points] (6 parts)

(a) What is the running time of RADIX-SORT on an array of n integers in the range $0, 1, \ldots, n^5 - 1$ when using base-10 representation? What is the running time when using a base-n representation?

(b) What is the running time of depth-first search, as a function of $|V|$ and $|E|$, if the input graph is represented by an adjacency matrix instead of an adjacency list?

(c) Consider the directed graph where vertices are reachable tic-tac-toe board positions and edges represent valid moves. What are the in-degree and out-degree of the following vertex? (It is O's turn.)

X	O	X
	O	
	X	

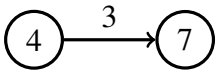
(d) If we modify the RELAX portion of the Bellman-Ford algorithm so that it updates $d[v]$ and $\pi[v]$ if $d[v] \geq d[u] + w(u, v)$ (instead of doing so only if $d[v]$ is strictly greater than $d[u] + w(u, v)$), does the resulting algorithm still produce correct shortest-path weights and a correct shortest-path tree? Justify your answer.

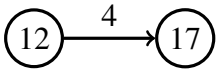
(e) If you take 6.851, you'll learn about a priority queue data structure that supports EXTRACT-MIN and DECREASE-KEY on integers in $\{0, 1, \ldots, u-1\}$ in $O(\lg \lg u)$ time per operation. What is the resulting running time of Dijkstra's algorithm on a weighted direct graph $G = (V, E, w)$ with edge weights in $\{0, 1, \ldots, W-1\}$?

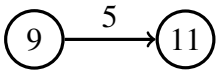
(f) Consider a weighted, directed acyclic graph $G = (V, E, w)$ in which edges that leave the source vertex s may have negative weights and all other edge weights are non-negative. Does Dijkstra's algorithm correctly compute the shortest-path weight $\delta(s, t)$ from s to every vertex t in this graph? Justify your answer.

Problem 3. You are the computer [12 points] (4 parts)

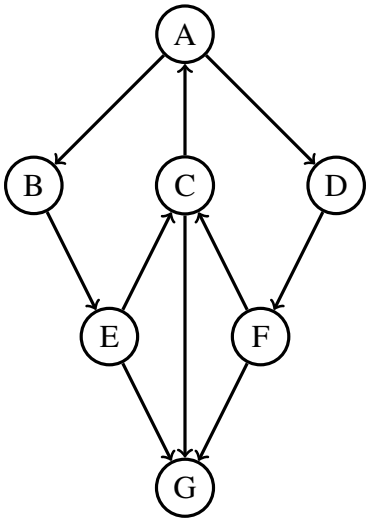
(a) What is the result of relaxing the following edges?

(i) 

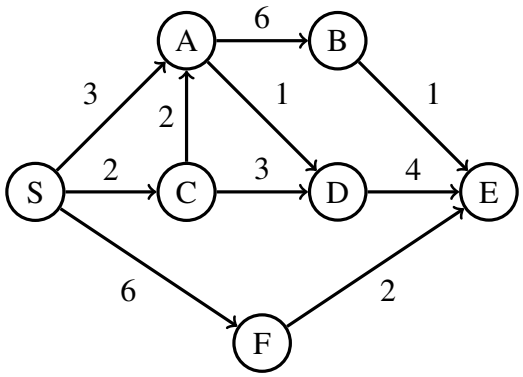
(ii) 

(iii) 

(b) Perform a depth-first search on the following graph starting at A. Label every edge in the graph with T if it's a tree edge, B if it's a back edge, F if it's a forward edge, and C if it's a cross edge. To ensure that your solution will be exactly the same as the staff solution, assume that whenever faced with a decision of which node to pick from a set of nodes, pick the node whose label occurs earliest in the alphabet.



(c) Run Dijkstra's algorithm on the following directed graph, starting at vertex S. What is the order in which vertices get removed from the priority queue? What is the resulting shortest-path tree?



(d) Radix sort the following list of integers in base 10 (smallest at top, largest at bottom). Show the resulting order after each run of counting sort.

Original list	First sort	Second sort	Third sort
583			
625			
682			
243			
745			
522			

Problem 4. Burgers would be great right about now [10 points]

Suppose that you want to get from vertex s to vertex t in an unweighted graph $G = (V, E)$, but you would like to stop by vertex u if it is possible to do so without increasing the length of your path by more than a factor of α.

Describe an efficient algorithm that would determine an optimal s-t path given your preference for stopping at u along the way if doing so is not prohibitively costly. (It should either return the shortest path from s to t or the shortest path from s to t containing u, depending on the situation.)

If it helps, imagine that there are burgers at u.

Problem 5. How I met your midterm [10 points]

Ted and Marshall are taking a roadtrip from Somerville to Vancouver (that's in Canada). Because it's a 52-hour drive, Ted and Marshall decide to switch off driving at each rest stop they visit; however, because Ted has a better sense of direction than Marshall, he should be driving both when they depart and when they arrive (to navigate the city streets).

Given a route map represented as a weighted undirected graph $G = (V, E, w)$ with positive edge weights, where vertices represent rest stops and edges represent routes between rest stops, devise an efficient algorithm to find a route (if possible) of minimum distance between Somerville and Vancouver such that Ted and Marshall alternate edges and Ted drives the first and last edge.

Problem 6. Just reverse the polarity already [10 points]

Professor Kirk has managed to get himself lost in his brand new starship. Furthermore, while boldly going places and meeting strange new, oddly humanoid aliens, his starship's engines have developed a strange problem: he can only make "transwarp jump" to solar systems at distance exactly 5 from his location.

Given a starmap represented as an unweighted undirected graph $G = (V, E)$, where vertices represent glorious new solar systems to explore and edges represent transwarp routes, devise an efficient algorithm to find a route (if possible) of minimum distance from Kirk's current location s to the location t representing Earth, that Kirk's ship will be able to follow. Please hurry—Professor Kirk doesn't want to miss his hot stardate!

Problem 7. The price is close enough [10 points]

As part of a new game show, contestants take turns making several integer guesses between 0 and $1{,}000{,}000$ (inclusive). In scoring each round, the show's host, Professor Piotrik Kellmaine, needs to know which two guesses were closest to each other. Provide an asymptotically time-optimal algorithm that answers this question, argue that it is correct, and give and explain its time complexity.

Problem 8. Call it the scenic route [10 points]

In the ***longest path problem***, we're given a weighted directed graph $G = (V, E, w)$, a source $s \in V$, and we're asked to find the longest simple path from s to every vertex in G. For a general graph, it's not known whether there exists a polynomial-time algorithm to solve this problem. If we restrict G to be acyclic, however, this problem can be solved in polynomial time. Give an efficient algorithm for finding the longest paths from s in a weighted directed acyclic graph G, give its runtime, and explain why your solution doesn't work when G is not acyclic.

Problem 9. Rated M for "Masochistic" [10 points]

You're playing the hit new platform video game, *Mega Meat Man*, and are having trouble getting through Level 6006. You've decided to model the level as a directed graph, where each vertex represents a platform you can reach, and each edge represents a jump you can try to make. After extensive experimentation, you've labeled each edge with the probability (a number in $[0, 1]$) that you can successfully make the jump. Unfortunately, if you fail to make any jump, you instantly die, and have to start over. Describe an efficient algorithm to find a path from the start platform s to the goal platform t that maximizes the probability of a successful traversal.

SCRATCH PAPER

SCRATCH PAPER