# Karatsuba's Algorithm

6.006 Review Session

# Problem Statement

- Given two n-digit long integers a and b in base r, find a × b.
- We've always assumed this is a constant time operation.
  - Makes life simpler.
  - Numbers are usually relatively small.
  - As we'll see, we can do multiplication relatively fast.

# Naïve Algorithm

- Using the algorithm we all love and know (the one we were taught in grade school) will take

$$O(n^2)$$

- Would like to improve on this…

# Divide and Conquer

- Let's try divide and conquer.
  - Divide each number into two halves.
    - $x = x_H\, r^{n/2} + x_L$
    - $y = y_H\, r^{n/2} + y_L$
  - Then:

$$xy = (x_H\, r^{n/2} + x_L)\, y_H\, r^{n/2} + y_L$$
$$= x_H y_H r^n + (x_H y_L + x_L y_H) r^{n/2} + x_L y_L$$

  - Runtime?
    - T(n) = 4 T(n/2) + O(n)
    - T(n) = O(n^2)

# Karatsuba's Insight

- Instead of 4 subproblems, we only need 3 (with the help of clever insight).
- Three subproblems:
  - $a = x_H\, y_H$
  - $d = x_L\, y_L$
  - $e = (x_H + x_L)\,(y_H + y_L) - a - d$
- Then $xy = a\, r^n + e\, r^{n/2} + d$
- $T(n) = 3\, T(n/2) + O(n)$
- $T(n) = O(n^{\log 3}) = O(n^{1.584\ldots})$

# Worked Example

- Compute 1234 * 4321.
- Subproblems:
  - $a_1 = 12 * 43$
  - $d_1 = 34 * 21$
  - $e_1 = (12 + 34) * (43 + 21) - a_1 - d_1$
    $= 46 * 64 - a_1 - d_1$
  - Need to recurse…

# Worked Example

- First subproblem:

$$a_1 = 12 * 43$$

- Subproblems:
  - $a_2 = 1 * 4 = 4$
  - $d_2 = 2 * 3 = 6$
  - $e_2 = (1+2)(4+3) - a2 - d2$
    $= 11$
- Answer: $4 * 10^2 + 11 * 10 + 6 = 516$

# Worked Example

- Second subproblem

$$d_1 = 34 * 21$$

- Subproblems:
  - $a_2 = 3 * 2 = 6$
  - $d_2 = 4 * 1 = 4$
  - $e_2 = (3+4)(2+1) - a2 - d2$
    $= 11$
- Answer: $6 * 10^2 + 11 * 10 + 4 = 714$

# Worked Example

- Third subproblem:

$$e_1 = 46 * 64 - a_1 - d_1$$

- Subproblems:
  - $a_2 = 4 * 6 = 24$
  - $d_2 = 6 * 4 = 24$
  - $e_2 = (4+6)(6+4) - a2 - d2$
    $= 52$
- Answer: $24 * 10^2 + 52 * 10 + 24 - 714 - 516$
  $= 1714$

# Worked Example

- Final Answer:

$$1234 * 4321 = 516 * 10^4 + 1714 * 10^2 + 714$$

$$= 5{,}332{,}114$$