Lecture 18: Shortest Paths IV - Speeding up Dijkstra

Lecture Overview

- Single-source single-target Dijkstra
- Bidirectional search
- Goal directed search - potentials and landmarks

Readings

Wagner paper on website, (upto Section 3.2)

DIJKSTRA single-source, single-target

```
Initialize()
Q ← V[G]
while Q ≠ φ
      do u ← EXTRACT_MIN(Q) (stop if u = t!)
      for each vertex v ϵ Adj[u]
          do RELAX(u, v, w)
```

Observation: If only shortest path from s to t is required, stop when t is removed from Q, i.e., when $u = t$

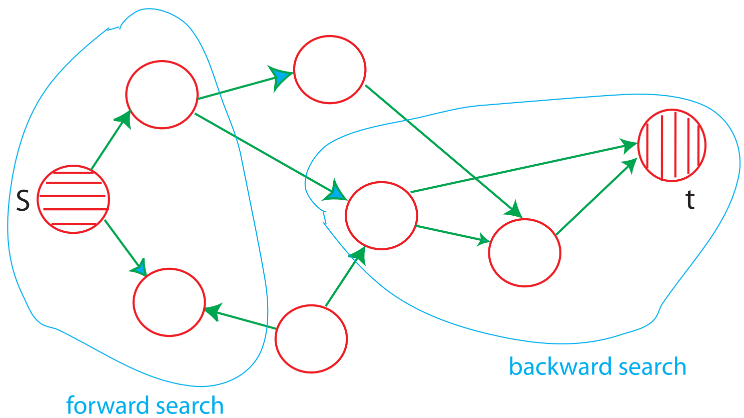

Figure 1: Bi-directional Search Idea.

Bi-Directional Search

Note: Speedup techniques covered here do not change worst-case behavior, but reduce the number of visited vertices in practice.

Bi-D Search

Alternate forward search from s
backward search from t
(follow edges backward)
$d_f(u)$ distances for forward search
$d_b(u)$ distances for backward search

Algorithm terminates when some vertex w has been processed, i.e., deleted from the queue of both searches, Q_f and Q_b
Subtlety: After search terminates, find node x with minimum value of $d_f(x) + d_b(x)$. x may not be the vertex w that caused termination as in example to the left!
Find shortest path from s to x using Π_f and shortest path backwards from t to x using Π_b. *Note:* x will have been deleted from either Q_f or Q_b or both.

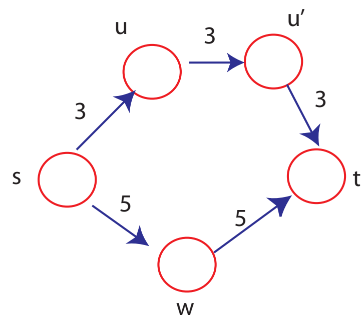

Figure 2: Bi-D Search Example.

Minimum value for $d_f(x) + d_b(x)$ over all vertices that have been processed in at least one search (see Figure 3):

$$d_f(u) + d_b(u) = 3 + 6 = 9$$

$$d_f(u') + d_b(u') = 6 + 3 = 9$$

$$d_f(w) + d_b(w) = 5 + 5 = 10$$

Goal-Directed Search or A^*

Modify edge weights with potential function over vertices.

$$\overline{w}\ (u, v) = w\ (u, v) - \lambda(u) + \lambda(v)$$

Search toward target as shown in Figure 4:

Correctness

$$\overline{w}(p) = w(p) - \lambda_t(s) + \lambda_t(t)$$

So shortest paths are maintained in modified graph with $\overline{w}$ weights (see Figure 5).

To apply Dijkstra, we need $\overline{w}(u, v) \geq 0$ for all (u, v).

Choose potential function appropriately, to be feasible.

Landmarks

Small set of landmarks LCV. For all $u \in V, l \in L$, pre-compute $\delta(u, l)$.

Potential $\lambda_t^{(l)}(u) = \delta(u, l) - \delta(t, l)$ for each l.

CLAIM: $\lambda_t^{(l)}$ is feasible.

Feasibility

$$\begin{aligned}
\overline{w}(u, v) &= w(u, v) - \lambda_t^{(l)}(u) + \lambda_t^{(l)}(v) \\
&= w(u, v) - \delta(u, l) + \delta(t, l) + \delta(v, l) - \delta(t, l) \\
&= w(u, v) - \delta(u, l) + \delta(v, l) \geq 0 \quad \text{by the } \Delta \text{ -inequality} \\
\lambda_t(u) &= \max_{l\, \epsilon\, L} \lambda_t^{(l)}(u) \ \ \text{is also feasible}
\end{aligned}$$

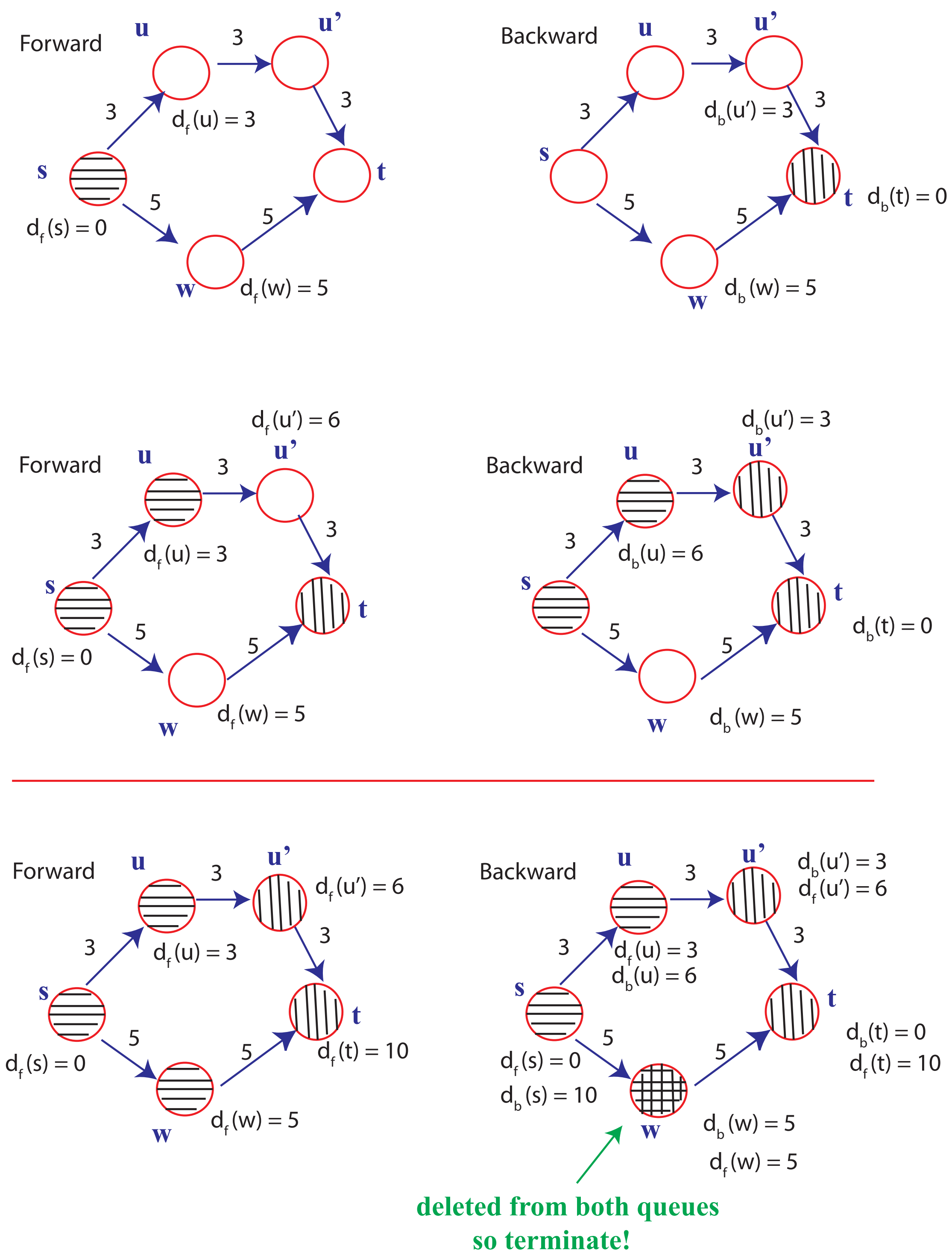

Figure 3: Forward and Backward Search and Termination.

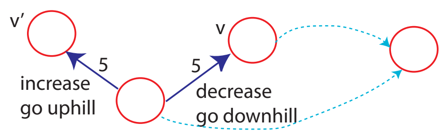

Figure 4: Targeted Search

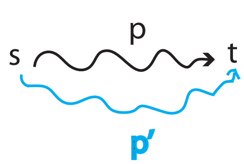

Figure 5: Modifying Edge Weights.