3 ways to interpret graphs:

- directed (arcs)
- undirected (edges)
- bidirected (darts)

Let $A$ be a finite set (arc set)
the set of darts is $A \times \{-1, 1\}$
$(a, 1)$ is often identified with the arc $a$.

A graph $G$ is a pair $(V, A)$ where $V$ is a partition of the dart set. That is, nodes are defined by their outgoing darts.

the tail of arc $a$ is the node $v \in V$
to which $(a, 1)$ belongs.

the head of arc $a$ is the node $v \in V$
to which $(a, -1)$ belongs.

we will sometimes denote arcs by their endpoints
\[ a = uv \] where $u(v)$ is $a$'s tail (head)
\[ u = (b, 1)(a, 1)(a, -1)(c, 1) \]

\[ \omega = (e, 1)(d, 1)(c, 1) \]

\[ v = (e, -1)(b, -1)(d, 1)(g, 1) \]

\[ x = (g, -1) \]

Note that our definition does not allow isolated nodes.

Define the bijection \( \text{rev}(e, \sigma) = (e, -\sigma) \).

The graph obtained from \((V, A)\) by deleting an edge set \( A' \) is \((V', A - A')\), where \( V' \) is the restriction of \( V \) to the darts in \( A - A' \).

Contracting an edge \( uv \in A \) from \((V, A)\) produces the graph \((V', A')\) where \( A' = A - \{uv\} \) and the parts \( u, v, s \) of \( V \) are merged (the darts of \( uv \) are removed).

Think about making a contracted edge shorter and shorter until its endpoints meet. Not well defined for self loops!
Embeddings: We will use combinatorial embeddings (aka rotation system)

An embedding of \( G = (V,A) \) is a permutation \( \pi \) on \( A \times \{ +1,-1 \} \) whose orbits (cycles) are exactly the nodes (parts) of \( V \).

[think of \( \pi \) as specifying, for every \( v \in V \), the darts whose tail is \( v \) in, say, counterclockwise order.]

An embedded graph is the pair \( G = (\pi,A) \) we will also use the notation \( G_\pi \)

Faces: define \( \pi^* = \pi \circ \text{rev} \)

the faces of \( G = (\pi,A) \) are the orbits of \( \pi^* \)

[When working with topological embeddings the faces of \( G \) are the connected components of the set of points in the sphere that are not assigned to any node or edge.]
e.g.:

\[ \pi = \left\{ (a, 1)(c, 1)(b, 1)(a, 1), (e, 1)(b, 1)(d, 1)(g, 1) \right\} \\
\left\{ (c, 1)(d, 1)(c, 1), (g, 1) \right\} \]

\[ \pi^* = \left\{ (b, 1)(d, 1)(c, 1), (d, 1)(g, 1)(g, 1)(e, 1) \right\} \\
\left\{ (e, 1)(b, 1)(a, 1)(c, 1), (a, 1) \right\} \]

Note: In our drawing, every face except \( \infty \) corresponds to a clockwise simple cycle. \( \infty \) is called the infinite face. Combinatorial embeddings do not distinguish infinite face (think of embedding on a sphere).
note: with combinatorial embeddings each connected component has its own infinite face

note: for connected graphs, combinatorial embeddings are equivalent to topological embeddings

**Dual graph:** the dual of $\mathcal{G} = (\pi, A)$ is the embedded graph $\mathcal{G}^* = (\pi^*, A)$

**Note:** when drawing $\mathcal{G}^*$ on $\mathcal{G}$, the order of darts in $\pi^*$ corresponds to clockwise order around dual nodes.

"Look at $\mathcal{G}^*$ from the other side of the paper"

$$\pi^* = \left( (b,1) (d,-1) (c,1) \right) \left( (d,1) (g,1) (g,-1) (e,-1) \right)$$

$$\left( (e,1) (b,-1) (a,1) (c,1) \right) \left( (a,-1) \right)$$
Lemma: the dual of the dual is the primal
\((G^*)^* = G\)

Proof: \((\pi^*)^* = (\pi \circ \text{rev}) \circ \text{rev} = \pi \square\)

How do combinatorial embeddings behave under deletions and contractions?

Deleting a dart \(d\) from \(G\) creates \(G'\),
where
\[\pi'(d') = \begin{cases} 
\pi \circ \pi(d') & \text{if } \pi(d') = d \\
\pi(d') & \text{otherwise}
\end{cases}\]

Lemma: contracting an edge of \(G\) is equivalent to deleting it in the dual \(G^*\)
(recall contraction is not defined for self loops)

intuition: in \(G^*\), \(u\) and \(v\) are faces, deleting the edge \(uv\) in \(G^*\) merges these two faces, so in \(G\), \(u\) and \(v\) are merged.
Proof: Let us be the edge to be deleted. Let \((a_0, a_1, \ldots, a_k)\) and \((b_0, b_1, \ldots, b_k)\) be the orbits of \(\pi\) that correspond to \(u\) and \(v\), respectively, and such that \(a_0\) and \(b_0\) are the darts of \(uv\). Since \(uv\) is not a self loop the two orbits are distinct. We want to show that \((\pi^*)^*\) is identical to \(\pi\) except that these two orbits are merged into \((a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k)\)

\[
(\pi^*)^* [d] = (\pi^*)^* [\text{rev}(d)]
\]

\[
= \begin{cases} 
\pi^* [\pi^* [\text{rev}(d)]] & \text{if } \pi^* [\text{rev}(d)] \text{ is deleted} \\
\pi^* [\text{rev}(d)] = \pi (d) & \text{otherwise}
\end{cases}
\]

Now, \(\pi^* [\text{rev}(a_k)] = \pi \circ \text{rev} \circ \text{rev}(a_k) = a_0\) and \(\pi^* [\text{rev}(b_k)] = \pi \circ \text{rev} \circ \text{rev}(b_k) = b_0\)

so \(a_k, b_k\) are the only two darts s.t. \(\pi^* [\text{rev}(d)]\) is deleted. Hence

\[
(\pi^*)^* [d] = \begin{cases} 
\pi^* [a_0] = \pi [b_0] = b, & \text{if } d = a_k \\
\pi^* [b_0] = \pi [a_0] = a, & \text{if } d = b_k \\
\pi [d] & \text{otherwise}
\end{cases}
\]

so \((a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k)\) is a new orbit after the deletion in the dual.
Planarity, we say that an embedding $\pi$ of $G=(V,A)$ is **planar** if it satisfies **Euler's formula**

$$n - m + f = 2K$$

- $n$: **# of nodes**
- $m$: **# of arcs**
- $f$: **# of faces**
- $K$: **# of connected components**

More generally, it can be shown that for any embedding $\pi$, $n - m + f = K (2 - 2g)$

$g$ is the **genus** of the embedding. Planar embeddings have $g=0$. 
you will prove in the PS that:

**Sparsity lemma:** for any planar embedded in which every face has size at least 3,

\[ m \leq 3n - 6 \]

implies no self loops & no parallel edges

**Interdigitating trees lemma:** Let \( T \) be a spanning tree of a planar embedded graph \( G = (\pi, A) \). The edges not in \( T \) form a spanning tree of \( G^* \).

Proof: (see draft of book for complete proof using combinatorial embeddings)

we first show that the edges not in \( T \) form a forest in \( G^* \).

Consider a cycle \( C \) in \( G^* \). By the Jordan curve theorem \( C \) partitions the sphere into two connected regions. Each of these regions contains at least one node of \( G \). Hence, considered as curves in the plane \( T \) crosses \( C \), so \( C \) contains at least one edge of \( T \). This implies that the subgraph induced by edges not in \( T \) is acyclic in \( G^* \), namely a forest.
It remains to show that the forest is a spanning tree. Since $T$ is a tree, $|T| = |V| - 1$.

By Euler’s formula $|V| - |A| + |V^*| = 2$.

Hence $|A| - |T| = |V^*| - 1$ so the forest is indeed a spanning tree. \( \square \)

For a spanning tree $T$ and a non-tree edge $e$, the fundamental cycle of $e$ with respect to $T$ is the cycle that consists of $e$ and of the unique simple path in $T$ between the endpoints of $e$. 
Cycle-Cut duality:

Recall that for a set $X$ of nodes, the edge cut $S(X)$ is called a bond (or a simple cut) if both sides of the cut are connected.

**Lemma:** Let $G=(\pi, A)$ be a planar graph. A set $C$ of edges is a simple cycle in $G$ iff it is a simple cut in $G^\pi$.

**Proof:** (again, see book for a proof that uses just combinatorial embeddings)

Let $C$ be a cycle in $G$. Let $X$ be the set of faces of $G$ (nodes of $G^\pi$) enclosed by $C$. By the Jordan curve theorem, any path between $X$ and $V^\pi \setminus X$ must cross $C$, so the edges of $C$ form a cut in $G^\pi$.

For any two dual nodes $f, g \in X$ there is a curve in the sphere that connects $f$ and $g$ and crosses no nodes of $G$. It follows that $f$ and $g$ are connected in the restriction of $G^\pi$ to $X$.

A symmetric argument holds for the restriction of $G^\pi$ to $V^\pi \setminus X$.  

□
Connectivity

Lemma: For any face $f$ of any embedded graph $G_x$, the darts comprising $f$ are connected.

Proof: Let $(d_0, d_1, \ldots, d_k)$ be the orbit of $\pi^*$ that corresponds to $f$. We show that $d_0, d_1, \ldots, d_k$ is a walk in $G$. For $1 \leq j \leq k$, $d_j = \pi^*(d_{j-1})$, so $\pi(\text{rev}(d_{j-1})) = d_j$, so $d_j$ and $\text{rev}(d_{j-1})$ have the same tail in $G_x$, so the head of $d_{j-1}$ is the tail of $d_j$. \qed

Connectivity Lemma: a set of darts forms a connected component in $G = (\pi, A)$ iff the same set forms a connected component in $G^*$. 

Proof: suppose $d, d'$ are connected in $G$, and let $d = d_0, d_1, d_2, \ldots, d_k = d'$ be a path of darts connecting them. For $i = 1, 2, \ldots, k$, the head of $d_{i-1}$ in $G$ is the tail of $d_i$. Thus, $d_i$ and $\text{rev}(d_{i-1})$ are in the same orbit of $\pi$, so are on the same face in $G^*$. Hence, $\text{rev}(d_{i-1})$ and $d_i$ are connected in $G^*$ and so are $d_{i-1}$ and $d_i$. 
Compression: We define compression of an edge as deleting it in the dual.

Already saw that compressing a non-self loop is contraction.
How about compressing self loops?

If \( e \) is a self loop in \( G \) then it is a cut-edge in \( G^* \). Deleting \( e \) from \( G^* \) makes it disconnected, so makes \( G \) disconnected as well.