Pleat folding: [Albers @ Bauhaus, 1927-1928]

"Hyperbolic paraboloid"  Circular pleat:

Self-folding origami: physics finds equilibrium form automatically
→ complex 3D shape from simple creases

Forces: paper wants to
- stay flat where uncreased
- stay bent where creased

Creasing = plastic deformation beyond yield point, changing elastic memory of paper to nonzero fold angle
Is the "hyperbolic paraboloid" really (approximating) a hyperbolic paraboloid?

**Surprise:** "hyperbolic paraboloid" doesn't exist! Impossible to nontrivially fold exactly that crease pattern \( \Rightarrow \text{fold angle} \neq 0, \pm 180^\circ \) [Demaine, Demaine, Hart, Price, Tachi 2009/2010]

**Good news:** (& likely what's happening in RL) possible to fold with extra creases & dropping a central crease:

**Proof:**
- Fold central \( \square \) hinge by some \( \Theta \)
- Next layer out determined by intersections of 3 known spheres:
- Intersection of 3 spheres can be computed by radical expression 
  \((±, ±, ±, \sqrt\) on center coords. & radii 
  (though >150,000 terms!)

- in theory, could specify exact 3D model 
  by such an expression

- in practice, infeasible beyond second ring

- instead: use interval arithmetic to 
  get coordinates within some range \([L, U]\)
- e.g. \([L_1, U_1] + [L_2, U_2] = [L_1+L_2, U_1+U_2]\)
- errors accumulate
- exists provided never take \(\sqrt{L \cdot U}\) 
  with \(L < 0\) (then sphere intersection 
  might not exist)
- also need to check no collision

- implemented in Mathematica
- checked for \(n=100\) rings 
  & \(\Theta \in \{2^\circ, 4^\circ, \ldots, 178^\circ\}\)

- required precision depends on \(n\) 
  (2048 decimal digits suffice for \(n=100\) 
  - nonalternating triangulation doesn't fold 
    to 180° (how much depends on \(n\))
So is it (approx.) a hyperbolic paraboloid?
- YES, very close, except at center
- great parabolic fit from just last 3 rings
- about 0.03% error at center
  (∼ independent of n)

OPEN: does triangulated folding exist for all n & 0 < θ < 180°?
- seems so, but lack tools to show

OPEN: does circular pleat exist?
- conjecture yes
How paper folds between creases:


- require folding to be piecewise-$C^2$ & flat
- crease = $C^1$ discontinuity
- semicrease = $C^2$ discontinuity
- flat = intrinsically flat (AKA “developable”)
  = zero curvature [Gauss’s Theorema Egregium]

Gaussian curvature at a point ($=0$)

= product of two principal curvatures

⇒ one is zero  $k_{\min} \& k_{\max}$ curvatures
- if both zero, planar point
- else parabolic point

Lemmas using differential geometry:

- every proper semicrease is a line segment with endpoints on creases or boundary

⇒ no semivertices (except on creases)
- every smooth point lies on a ruled segment with endpoints on creases or boundary, unique unless point has a planar neighborhood

$\Rightarrow$ ruled surface $c(s)+t \cdot S(s)$ around any point

$C^1 \cup C^0$

- torsal: common tangent plane to each rule line
- points along rule line uniformly planar/parabolic

skip
Polyagonal $\Rightarrow$ flat: if uncreased region's boundary folds to a 3D polygon, then entire region folds to a 3D plane.

Proof: claim every point in region is planar
- consider parabolic point $p$
- a small neighborhood of $p$
  is entirely parabolic (by smoothness)
- take union of rule lines through those points
- $ac$ & $bd$ polygonal
- look at segment $bf$ of $bd$
- $n(q)$ = normal at $q$ on $bf$
  is perpendicular to $bf$
  & to $q$'s rule line
- torsal $\Rightarrow$ same normal along rule line
  $\Rightarrow n'(q)$ is perpendicular to rule line
  derivative as $q$ moves along $bf$
- also perp. to $bf$ because all $n(q)$ are
  $n'(q)$ has same direction as $n(q)$
- $n'(q) = 0$
- $n(q)$ constant
- rule lines form planar region

$\square$
Straight creases stay straight:
geodesic crease with fold angle $\neq \pm 180^\circ$
folds to 3D line segment

Proof: consider point $p$ interior to crease
- surrounded by 2 sides
- compute tangent plane
  on each side: $Sp, Tp$
- different because fold angle $\neq \pm 180^\circ$
- tangent vector $p'$ along crease
  lies along $Sp \cap Tp$
- consider curvature vector $p''$
- crease is straight on unfolded paper & paper is locally flat $Sp$ & $Tp$
$\Rightarrow$ crease should have zero curvature when projected onto $Sp$ or $Tp$
$\Rightarrow$ $p''$ is perpendicular to $Sp$ & $Tp$
- $Sp \neq Tp \Rightarrow p'' = 0$
$\Rightarrow$ crease is a line segment. □

Nontrivial foldings of straight creases are $\approx$ rigid!
every interior face of straight crease pattern
is not touching boundary of paper
has polygonal boundary ($creases \rightarrow segments$)
& thus is planar in 3D i.e. rigid
(boundary faces might not be rigid)
Back to “hyperbolic paraboloids”:

Center is bad: \( \square \cong \square \) rigid
- can fold one crease but not both
  (for nontrivial folding)

Any ring is bad:

\[ \Rightarrow \text{induces folding of} \quad \square \quad \text{again} \]

More generally: two rings bad
- if diagonal extensions meet
  - must have local mountain/valley assignments like
    (or reverse)

\[ \Rightarrow \text{diagonals in ring are all} \ M \ \text{or all} \ V \]
- induces all-M or all-V folding of
- impossible
OPEN: what is the maximum volume whose surface is a folding of a teabag
doubly covered square

\[ \text{e.g., } \square \Rightarrow \star \Rightarrow 4\text{-hypar} \]

Inflation:
- every generic convex polyhedron can be folded to increase volume \[ \text{[Bleecker 1996]} \]
  “by simultaneously delivering karate chops to the edges of the polyhedron”

\[ \text{\quad valley} \]

- every polyhedron can be folded to increase volume \[ \text{[Pak 2006]} \]
\[ \Rightarrow \text{limit not polyhedral} \]

Curved creases \[ \text{[Huffman 1970s - 1990s]} \]
\[ \text{[Demaine, Demaine, Koschitz 2010]} \]
\[ + \text{ Huffman family} \]