Pocket flipping: closed chains with universal joints are flat-state connected (follows from Carpenter's Rule "alternate approach")

Pocket of 2D polygon = region outside polygon & inside convex hull

Pocket lid = convex-hull edge

Flip = reflect pocket through its lid
  = rotate 180° through 3D around the lid
  - avoids self-intersection (line of support)
  - increases area

"Erdős-Nagy" Theorem: [posed by Erdős 1935]
  any polygon always convexifies after finite flips, no matter how flip sequence is chosen
  - but can be arbitrarily many: [Joss & Shannon 1973]
  - OPEN: bound # flips in n & r = max. dist./min. dist pseudopolynomial? [Overmars 1998]
"Proofs" of Erdős-Nagy Theorem: [Demaine, Gassend, O'Rourke, Toussaint 2007]

Nagy 1939
- flawed: "P₀ ≤ C₀ ≤ P₁ ≤ C₁ ≤ ..."
  (used to "prove" limit polygon convex)

Reshetnyak 1957
- correct (though somewhat imprecise)

Yusupov 1957
- flawed: "limit convex else flip"
  & more subtle error

Bing & Kazarinoff 1959
- correct (though somewhat terse)

Wegner 1993
- flawed: "move vertex ⇒ increase area by increment Δ"

Grünbaum 1995
- omission: why limit polygon is convex

Toussaint 1999/2005
- flawed: "limit convex else flip"

Demaine, Gassend, O'Rourke, Toussaint 2008
- generalization to self-crossing assuming no "hairpins"

consider an infinite flip sequence

1. distance from a vertex to fixed point \( x \) inside the polygon (remains so) only increases
   - pocket lid is Voronoi diagram of old & new

2. each vertex approaches a unique limit
   - apply 1 to three noncollinear points \( x_1, x_2, x_3 \) inside the polygon
   - distances from vertex \( \leq \) perimeter of polygon/2
   \( \Rightarrow \) distances converge
   \( \Rightarrow \) vertex approaches intersection of 3 circles

3. turn angle at each vertex converges
   - by 2, 3 vertices defining the angle converge
   - by 1, vertices do not get closer to each other
   - rest by continuity

4. vertex moves infinitely \( \Rightarrow \) asymptotically flat
   - each move negates sign of turn angle \( \Rightarrow \) \( \rightarrow 0 \)

5. contradiction
   - eventually asymptotically pointed vs. stop moving
   \( \Rightarrow \) attain limit convex hull, but about to flip! \( \square \)
Flipturn: rotate pocket 180° in 2D around lid midpoint
- at most \( n! \) configurations [Joss & Shannon 1973]
- always \( O(n^2) \) flipturns [Aichholzer et al. 2002; Ahn et al. 2000 (diff. model)]
- sometimes \( \Omega(n^2) \) flipturns [Biedl 2004]
- final polygon & location determined
- NP-hard to find longest flipturn sequence
- \( \boxed{\text{OPEN}} \): finding shortest flipturn sequence?

Orthogonal polygons: \( < n \) flipturns
- count brackets: [ & ] (polygon interior on either side)
  - allow overlap \( [] \) \( \Rightarrow \leq n \) brackets
- claim # brackets never decreases (13-case analysis)
- orthogonal flipturn kills two brackets:

\( \Rightarrow \leq \frac{n}{2} \) orthogonal flipturns
- diagonal flipturn kills two vertices:
  \( \Rightarrow < \frac{n}{2} \) diagonal flipturns
  \( \Rightarrow < n \) total

\( \boxed{\text{OPEN}} \): \( n - O(1) \) flipturns ever possible?
- best example requires \( \frac{5}{6} n - O(1) \)
Flipturns: (cont'd)

General polygons: \( \leq ns \) if \( s \) distinct slopes
- discrete turn angle = \( 1 + \# \) slopes between
- measure total discrete turn angle:
  - nondegenerate flipturn decreases by \( \geq 2 \)
  - degenerate flipturn doesn't change
- also count brackets:
  - nondegenerate flipturn increases by \( \leq 2 \)
  - degenerate flipturn decreases by \( \geq 2 \)
- potential function = total disc. angle + \( \frac{1}{2} \) \# brackets
- any flipturn decreases by \( \geq 1 \)
- initially \( \leq n(s-1) + n = ns \)
Deflation: inverse of flip (avoiding crossings)
- conjectured finite \[\text{[Wegner 1993]}\]
- quadrilaterals with \(a+c = b+d \land a \neq b+c+d=a\) always deflate infinitely
  \[\text{[Fevens, Hernandez, Mesa, Morin, Soss, Toussaint 2001]}\]
- that's all such quads \[\text{[Ballinger, PhD 2003]}\]
- no pentagon always deflates infinitely
  \[\text{[Demaine, Demaine, Fevens, Mesa, Soss, Souvaine, Taslakian, Toussaint 2007]}\]
- **OPEN**: any \(n \geq 6\) gon with no flat vertices that always deflates to flat limit?
- any infinite deflation sequence has a unique limit polygon \[\text{[Hubard & Taslakian 2010]}\]
- **OPEN**: characterize infinitely deflectable polygons - algorithm for testing a sequence?
Pop: flip on 2 incident edges
- can be forced to introduce crossings ⇒ allow
- possible to convexify any polygon in finitely many pops? [Ballinger & Thurston 2001]
- NO: “alternating” polygons can’t be [Dumitrescu & Hilscher 2009]
  vertices alternate between x & y axes
  ⇒ preserved under pops
  ⇒ never convex for n > 4

Popturns: flipturn on 2 incident edges
- equivalent to pops for equilateral polygons
- can convexify any polygon allowing self-intersection [{Aloupis et al. 2007]
Linkages in 4D: [Cocan & O'Rourke 2001]
- every open chain can be straightened in 4D:
  - idea: move first bar to “extend” second bar
  - then “fuse” that joint,
    treating first two bars as one
  \[\Rightarrow\] effectively n-1 bars left; induct
- problem: goal state for first bar might intersect rest of chain
  - if so, just perturb the linkage (actually can just move the vertex to be straightened)
- key: first bar can reach any nonobstructed position
- configurations around joint
  = points on 3D sphere in 4D
  (centered at joint)
  (analogy: 3D chain, points on usual 2D sphere)
- obstacle = projection of 1D bar onto sphere
  = 1D arc
- deleting 1D arcs keeps 3D sphere connected
  (analogy: deleting 0D points from usual 2D sphere)
- every tree can be flattened
- similar technique
- every cycle can be convexified (diff. approach)