Folding polyhedra:
Decision problem: given a polygon (or connected metric polygonal 2-manifold), can its boundary be glued to itself (in pairs of intervals) such that resulting surface can be folded into exactly a convex polyhedron? (no multiple layers like origami)

Enumeration problem: list all gluings & foldings
Combinatorial problem: how many can there be?

Why convex polyhedra? Always possible to fold into a (nonconvex) polyhedron provided orientable or some unglued boundary.

[Burago & Zalgaller 1960, 1996; O'Rourke 2010]

Alexandrov gluing: polygon + gluing induce a metric by shortest-path lengths between all pairs of points

- metric is polyhedral: all but finitely many points have zero curvature
- metric is convex if all points have zero or positive curvature
- metric is topological sphere if gluing noncrossing shortest paths from x to all vxs
Alexandrov's Theorem: [1941; English book 2005]
every convex polyhedral metric, topologically a sphere,
is realized by a unique convex polyhedron
(possibly degenerating to doubly covered flat polygon)

Proof sketch: [Lecture 14]

Uniqueness: draw all shortest paths between pairs of vs.
- includes all edges of any polyhedral realization
  ⇒ faces between mesh of paths are rigid
- Cauchy’s Rigidity Theorem ⇒ unique convex realiz.

Existence: induct on \( n = \#\) vertices
- base case: \( n \leq 4 \) (double triangle or tetrahedron)
- total curvature of all vertices = \( 720° = 4\pi \)
  [Descartes’ theorem; consq. of Gauss-Bonnet Formula]
- \( n \geq 5 \) ⇒ 2 vertices \( x, y \) have curvatures \( \alpha, \beta < 180° \)
- along shortest path from \( x \) to \( y \), paste edge of a doubly covered triangle
  ⇒ new vertex @ triangle apex; adds material @ \( x \& y \)
- continuously vary angles of triangle at \( x \& y \)
  from \( 0 \) to \( \frac{\alpha}{2} \& \frac{\beta}{2} \) ⇒ \( x \& y \) flatten
  ⇒ continuous path on manifold of metrics
  from original metric to metric with one less vertex
- induct on latter
- argue continuity of realizability using Implicit Function Theorem ⇒ nonconstructive ∎
Constructive Alexandrov’s Theorem: [Bobenko & Izmestiev 2006]
(following Blaschke & Herglotz 1937; Alexandrov 1950; Volkov 1955)

Idea: represent interior of polytope, not just boundary
- add (hypothetical) point \( p \) interior to polytope
- triangulate surface with geodesics
- form solid tetrahedron on \( p \) & each \( \Delta \)
- solve for distance \( r_i \) from \( p \) to vertex \( v_i \)
  \( \Rightarrow \) determines geometry of tetrahedra, hence polytope

Generalized polytope: same combinatorial structure, tetrahedra glued around \( p \), but not nec. in 3D
- consider dihedral angles of edges of tetrahedra \( \sim \) view as angle of solid material
- convexity invariant: \( \Sigma \) two dihedral angles incident to edge of surface triangulation \( \leq 180^\circ \)
- goal: reach real polytope where \( X_i = 360^\circ - \Sigma \) dihedral angles around interior edge \( (p, v_i) = 0 \)

Evolution: start at generalized polyhedron \( P(0) \)
- set \( X_i(t) = (1-t)X_i(0) \rightarrow 0 \) as \( t \rightarrow 1 \)
- differential equation to evolve \( r_i \)'s:
  \[
  \frac{d r_i}{d t} = (\frac{\partial^2}{\partial p_i^2})^{-1} \cdot \overline{p} (\bar{0})
  \]
  Jacobian - how \( r_i \)'s affect \( X_j \)'s
- geodesic triangulation changes (flips) as \( t \rightarrow 1 \)
- crucial part: Jacobian non-zero & has inverse (uses inverse function theorem!)
**Constructive Alexandrov's Theorem: (cont'd)**

- **Starting point:** need a generalized polyhedron P(\(\emptyset\))
  - geodesic Delaunay triangulation of surface
  - setting all \(r_i\) equal & sufficiently large yields desired convexity invariant
    - using Delaunay property

**Pseudopolynomial algorithm for Alexandrov's Theorem:**

\[ O(n^{456.5 r^{1891/3^{121}}} ) \text{ time} \]

\[ \Rightarrow \text{accuracy} \]
\[ \Rightarrow \text{spread} = \text{largest dist./smallest dist.} \]

- compute geodesic Delaunay by modifying
  - [Mitchell, Mount, Papadimitriou 1987]
  - to handle when edges not nec. shortest paths
- make each part effective with explicit bounds:
  - how large to make initial \(r_i\)’s
  - Jacobian & inverse bounded away from \(\emptyset\)
    (using Hessian instead of inverse function thm)

**OPEN:** polynomial time possible?

- logarithmic dependence on \(r/\varepsilon\) possible:
  - reduces to roots of \(2^{\Theta(n)}\)-degree polynomial

  [Sabitov 1996; Fedorchuk & Pak 2005]
**Ungluable polygon**: [Demaine, Demaine, Lubiw, O'Rourke 2000]

- no vertex can be glued into red reflex vertex: $< 90^\circ$ free

  - “zip” red reflex vertex
  - green reflex vertices glued together
  - $> 360^\circ$ of material

**Random polygons are ungluable:**

- suppose uniform distribution on angles & edge lengths

  - $\approx \frac{1}{2}$ reflex vertices

- gluing in a convex vertex still leaves reflex vertex (angles don't match)

- at some point must zip a reflex vertex

  - fails if nearer angle is reflex:

    - convex \[\Rightarrow OK\]
    - reflex \[\Rightarrow BAD\]

- happens with probability $\frac{1}{2}$ for each reflex vertex
Perimeter halving: every convex polygon has an Alexandrov gluing
- pick any point \( x \) on polygon boundary
- glue together two boundary points at distance \( d \) from \( x \) (measured along boundary), for all \( d \geq 0 \)
- both points have \( \leq 180^\circ \) of material \( \Rightarrow \) convex
- stop at diametrically opposite point \( y \)
\( \Rightarrow \) gluing two halves (paths) of perimeter from \( x \) to \( y \)
- \( x \) & \( y \) also convex (nothing glued)
\( \Rightarrow \) Alexandrov

**EXPERIMENT:** cut out convex polygon
tape together perimeter halves
see what convex polyhedron you get

 Mostly different: uncountably many polyhedra
- vary \( x \) near vertex \( v_i \), say \( d \) along edge \( v_i v_{i+1} \)
- \( x \) & \( v_i \) become distinct vertices of
shortest-path distance \( d \)
- only finitely many vertex-vertex shortest paths
for a particular polyhedron
- uncountably many choices for \( d \)
\( \Rightarrow \) uncountably many polyhedra
Gluing tree:
- turn polygon “inside-out”
- gluing of that boundary to self forms a cycle around a tree
- corresponds to cutting tree in unfolding

Properties:
- each leaf is either a zipped vertex or a fold point in middle of edge ($\Rightarrow 180^\circ$)
  $\Rightarrow$ at most 4 fold points ($720^\circ$ total curvature)
- if 4 fold points, then these are only leaves
  $\Rightarrow$ always induce curvature
- at most one nonvertex (middle of edge) glued at $\geq 3$-way junction (else $180^\circ \cdot 2 + \text{something}$)
Rolling belt = path in gluing tree whose end points are either fold pts. or convex vx. leaves & along which always ≤ 180° material on either side = effectively an embedded convex polygon 
⇒ can perimeter halve arbitrarily = “rolling the belt”
⇒ only way to get infinite gluings

Examples:
1 rolling belt: perimeter halving of convex polygon
cylinder

2 rolling belts: 

3 rolling belts:

≈ 4 rolling belts: impossible [6.885 Fall 2004 PS5: 3] 
⇒ must be 4 fold points
⇒ no curvature elsewhere
⇒ rolling belt from one fold point is uniquely determined to some fold point
⇒ same rolling belt from latter fold point
⇒ ≤ 2 rolling belts