Polyhedron (Un)Folding:

Folding: when can a polygon be glued along its boundary to form (exactly) a convex polyhedron? (only one layer allowed, unlike origami)

Unfolding: when can a polyhedral surface be cut & unfolded into one nonoverlapping planar piece?

Edge unfolding: just cut along polyhedron's edges
General unfolding: can cut interior to faces

Summary:
- convex polyhedra
  - edge unfolding: OPEN
  - general unfolding: ALWAYS
- non-convex polyhedra
  - edge unfolding: NOT ALWAYS
  - general unfolding: OPEN

Big questions:

OPEN: does every convex polyhedron have an edge unfolding? \([\text{Dürer 1525; Shephard 1975}]\)

OPEN: does every polyhedron without boundary have a general unfolding? \([\text{Bern, Demaine, Eppstein, Kuo 1999}]\)
Curvature of a vertex = $360^\circ - \Sigma$ incident face angles
- positive $\Rightarrow$ convex cone
- zero $\Rightarrow$ flat
- negative $\Rightarrow$ saddle

Cutting = cuts in a valid unfolding
- only zero-curvature vertices can be flattened without cutting or local overlap
$\Rightarrow$ any cutting spans all nonzero-curvature vertices
- indeed, if curvature $< -k \cdot 360^\circ$
  then cutting must have degree $> k + 1$
- if polyhedron has no handles (sphere/disk, not torus)
  then cutting has no cycles (else $> 1$ piece)
  $\Rightarrow$ spanning forest
- connected component of cutting makes boundary component of unfolding
$\Rightarrow$ if polyhedron has no boundary or handles
  & unfolding has no holes
  then cutting is a spanning tree
  $-$ cf.:
  $\Rightarrow$

- if polyhedron is convex
  then cutting is a spanning tree
Trivial bad example: [Bern, Demaine, Eppstein, Kuo 1999]

- polyhedron with boundary & just one vertex, of negative curvature
  - need $\geq 2$ cuts at vertex
  - can't stop cutting until we reach the boundary (else could reglue cuts without change)

$\Rightarrow$ disconnect surface
$\Rightarrow$ no general unfolding

Shortest path between two points $x$ & $y$ on polyhedron
  - unfolds straight (geodesic)
  - doesn't cross itself
  - doesn't pass through a positive-curvature vertex
General unfoldings of convex polyhedra:

Star of shortest paths from point \( x \) to all other points
- if two shortest paths touch beyond \( x \)
  then either one is a subpath of another
  or they touch only at their ends
  \( \Rightarrow \) nonunique shortest path

Cut locus / ridge tree with respect to point \( x \)
= points with nonunique shortest paths from \( x \)
= Voronoi diagram of \( x \)
- spanning tree of polyhedron
- leaves = the polyhedron vertices

Source unfolding [Sharir & Schorr 1986; Mount 1985; Mitchell, Mount, Papadimitriou 1987]
- cut along the cut locus
- unfold star of shortest paths from \( x \)
\Rightarrow \) star-shaped unfolding: boundary visible from \( x \)

Star unfolding [Alexandrov 1948; Aronov & O’Rourke 1992]
- cut along shortest paths from (generic) point \( x \)
  to every polyhedron vertex (star of cuts)
- much harder to prove nonoverlap
General unfoldings of convex polyhedra: (cont’d)

Extensions:
- both source & star unfoldings generalize to x=geodesic path [Itoh, O’Rourke, Vilcu 2008, 2009]
- neither works for nonconvex polyhedra
- source unfolding works in higher dimensions [Miller & Pak 2003]
- source unfolding can be “continuously bloomed” without intersection [Demaine, Demaine, Hart, Iacono, Langerman, O’Rourke 2009/2010]
- also, any unfolding can be refined to be continuously bloomable
- **OPEN**: true of star unfolding?
  all edge/general unfoldings?
- **OPEN**: other general unfoldings?
Edge-unfolding convex polyhedra:
- implicitly dates back to Albrecht Dürer’s Painter’s Manual [1525]
- possible for every example we’ve tried
  - e.g. Archimedean
  - heuristic/exhaustive search: commercial software, JavaView Unfold, JavaGami, Unfold for Blender, Pepakura Designer [http://www.tamasoft.co.jp.pepakura-en/]
  - Schlickenreider [1997] search
- all efficient algorithms we’ve tried fail [Schlickenreider 1997; Lucier 2006]
- some simple examples overlap
  - e.g. sliver tetrahedron
- random cutting of random convex polyhedron overlaps with probability $\to 1$ as $n \to \infty$ [Schenvon & O’Rourke 1987] hull of rand. pts. on sphere
- OPEN: prove this empirical observation

An approach: [Bern, Demaine, Rote, G.Price, ...]
- OPEN: edge-unfold convex terrains (project to a plane without intersection)
  - positive equilibrium stress
- OPEN: edge-unfold “almost flat” terrain/polyhedron (scale $z \to \varepsilon z$, $\varepsilon \to$ infinitesimal)
  - visible in plane
- challenging even for prisms/convex hull of two parallel polygons
Edge-unfolding convex polyhedra: (cont’d)

Solved special classes:

- ≤ 6 vertices [DiBiase 1990]

- pyramid = convex hull of convex polygon + point

- prism = convex hull of convex polygon + parallel offset

- prismoid = convex hull of two parallel convex polygons with matching angles [O’Rourke 2001] Volcano again

- dome = all faces share edge with single base [O’Rourke–GFALOP] Volcano again

- OPEN: prismatoids = convex hull of two parallel convex polygons
  - possible in “smooth” case [Benbernou, Cahn, O’Rourke 2004]
  - band of side faces unfolds [Aloupis 2005]
Fewest nets: edge-unfold convex polyhedron into a "small" number of pieces
- want to know whether 1 is possible
- $F = \# \text{ faces}$ is trivial (cut out each)
- $\frac{2}{3} F$ by pairing together $\frac{2}{3}$ of faces [Spriggs 2003]
- $\frac{1}{2} F$ by fancier argument [Dujmović, Morin, Wood 2004]
- better bounds [Pinciu 2007]
- OPEN: $o(F)$ possible?

Edge-unfolding nonconvex polyhedra:
- trivial unfoldable example: insufficient area in donut hole [Biedl et al. 1999]
- with all faces ~ disks: can't connect two X's [Biedl et al. 1999]
- with all faces triangles $\Rightarrow$ two share only one edge ("topologically convex") [Bern, Demaine, Eppstein, Kuo, Mantler, Snoeyink 2003]
- smaller examples [Grüenbaum 2001 & 2002]
Triangulated un unfolded example:
- suppose base vertices of spike have neg. curvature, even without one spike \( \Delta \)
  (brim angle = \(300^\circ - \varepsilon\),
  spike angle = \(90^\circ - \varepsilon \Rightarrow 390^\circ - 2\varepsilon\))
- claim: can't edge-unfold a hat by itself
  - spanning forest has \(\geq 2\) leaves
  - can't be at negative curvature vertices
  - can't have two on boundary
  - one at peak, one on boundary
  - two possibilities remain
  - both leave all but one spike \( \Delta \) at a base vertex of spike
  \(\Rightarrow\) must be a path of cuts between two boundary vertices, interior to hat
- these 4 paths force cycle on 4 vertices

\(\Rightarrow\) no one-piece edge unfolding \(\square\)