Locked linkages: recall

- 2D chains never locked
- 3D can lock
- 4D+ never locked

Algorithms for unfolding 2D chains

1. ordinary differential equation given by (canonical) expansive infinitesimal motion \( \frac{dE}{dt} = d \) (Connelly, Demaine, Rote 2000, 2002)
   - strictly expansive (other than
   - one step in poly. time: convex program
   - many steps: inaccurate (without projection)
     OPEN: how many? pseudopolynomial?

2. pointed pseudotriangulations (Streinu 2000, 2005)
   - expansive \( \Rightarrow \) maximal edge set on given points with \( >180^\circ \) angle at every vertex
   - \( n^{O(1)} \) steps
   - one step follows 1D.O.F. linkage \( \Rightarrow \) delete edge of convex hull
   - best algorithm is exponential
   - OPEN: are pseudotriangulations easier than general 2D linkages? (e.g., they are noncrossing)
   - PROJECT: implement this algorithm
Algorithms for unfolding 2D chains: (cont’d)

3 energy

- not expansive
- one step is $O(n^2)$ & exact on real RAM
- pseudopolynomial number of steps
  poly. in $n$ & $r = \max\dist/\min\dist$

**Approach:**

- define energy function on configurations:
  \[ E(C) = \sum_{\text{edge } vw} \sum_{\text{vertex } u \neq v \text{ or } w} \frac{1}{d(u,vw)} \]

- any energy-decreasing motion avoids crossings: approaching $O\dist$ shoots $E \to \infty$
- expansive motion decreases energy
  (in fact, every term)

$\Rightarrow$ energy-decreasing motions exist

- downhill gradient of energy exists: $-\nabla E$
  - computable in $O(n^3)$ time
- lower bound gradient, upper bound curvature

$\Rightarrow O(n^{1.23} r^{1.1})$ step bound (!)

**OPEN:** improve step bound (likely not hard)

**OPEN:** $n^{O(1)}$ step bound possible? Conjecture no

**OPEN:** is minimum-energy configuration unique?
  For equilateral polygons, it’s a regular $n$-gon
Single-vertex rigid origami: [Streinu & Whiteley 2001]

Every folded state of a single-vertex crease pattern can be folded rigidly (continuously, faces staying rigid)

\[ \square = \bigcirc \]

Linkage folding!

Spherical Carpenter's Rule Theorem: [Streinu & Whiteley]

A closed chain of total length \( \leq 2\pi \) on unit sphere has a connected configuration space.

- Proof based on projective invariance of infinitesimal rigidity.
- Length \( \leq 2\pi \) \( \Rightarrow \) lie in hemisphere \( \Rightarrow \) can project to plane.
- Length \( > 2\pi \) \( \Rightarrow \) no convex configuration.

Touching case (e.g., flat folding) handled by recent self-touching Carpenter's Rule Theorem [Abbott, Demaine, Gassend 2007]
Locked 2D trees:

- deg. $\geq 5$
- diameter 4
- only 1 degree-3 vertex

[Biedl et al. 1998/2002] [Poon 2005] [Demaine, Rote 2002]

V

- 8 edges
- linear

H

- orthogonal

- equilateral
- not tight

[Ballinger, Charlton, Demaine, Demaine, Iacono, Liu, Poon 2009]

- linear = edges lie (nearly) in a line
- locked linear trees have
  - $\geq 8$ edges
  - $\geq 9$ edges or diameter $\geq 6$

[Ballinger et al.]

OPEN: 8 edges minimal for nonlinear?
14 edges minimal for orthogonal?
**OPEN**

- characterize locked linkages
e.g. locked trees in 2D or chains in 3D
  - polynomially solvable?
  - special case: linear trees

**Related problem:** can you fold config. A $\rightarrow$ config. B?
  - PSPACE-complete for 2D trees & 3D chains
    - [Alt, Knauer, Rote, Whitesides 2004]
  - but their reductions use locked linkages as gadgets — so all locked
Infinitesimally locked linkages [Connelly, Demaine, Rote 2002]

Intuition: in many locked examples (particularly 2D), as gaps get smaller, so do valid motions

Locked within $\varepsilon$ = configuration from which it is impossible to get farther than $\varepsilon$

Rigid = locked within $\varnothing$ - but trees are never rigid... right?

Self-touching configuration allows infinitesimal gaps: geometric overlap, distinguished combinatorially

- now can be rigid

Return to nontouching: rigidity $\Rightarrow$ "strongly locked"

Strongly locked = sufficiently small perturbations are locked within $\varepsilon$, for any $\varepsilon > 0$

$S$-perturbation = move vertices within $S$-disks, preserving combinatorial sidedness

Every self-touching has a (non-self-touching) $S$-perturbation, for all $S > 0$ [Ribó Mor, PhD 2006]
Proof based on "slippery rigidity": [Connelly 1982]
if relax the edges in a rigid tensegrity
(bars can change length by \( \varepsilon \)
struts can shrink by \( \varepsilon \), etc.)
then still can't move more than \( \varepsilon \)

Infinitesimally locked linkages: (cont’d)

\[\text{Infinitesimal rigidity:}\]
- implies rigidity
- "zero-length strut" (linear inequality):
  \( u \) should remain right of \( uv \)
- sometimes nonconvex:

\[\Rightarrow \text{conservative polynomial test (drop constraints)}\]
or exponential test (split into 2 convex)
- analogs of equilibrium stress & duality
- even Maxwell-Cremona [Ribó Mor. PhD 2006]
- nice proofs by hand: positive stress on struts
  + underlying linkage rigid
  \( \Rightarrow \text{inf. rigid} \)
  \( \Rightarrow \text{rigid} \)
  \( \Rightarrow \text{strongly locked} \)

PROJECT: implement locked linkage tester/designer tool
Infinitesimal locking rules: [Connelly, Demaine, Demaine, Fekete, Langerman, Mitchell, Ribó, Rote 2006]

**Rule 1:**

- acute
- equal length
- pinned together for positive time

**Rule 2:**

- acute
- equal length
- ditto

**Example:**

```
V \rightarrow [Rule 1] \rightarrow [Rule 1] \rightarrow [Rule 2]
```

- rigid
- strongly locked
3D knitting needles: locked if each end bar is longer than $\Sigma_i$ middle bars

\[ \text{[Cantarella & Johnston 1998]} \]

Proof: draw ball $B$ centered at midpoint of middle bars, diameter $= \Sigma_i$ middle bars $+ \varepsilon$

$\Rightarrow$ middle vertices remain inside $B$,
end vertices remain outside $B$.

$\Rightarrow$ any motion could be augmented by an unknotted rope connecting two ends outside $B$.

$\Rightarrow$ straightening motion would untie trefoil knot. $\Box$

OPEN: minimum possible edge length ratio for which locked 3D chain exists?
- best example is $1:3+\varepsilon$ above

OPEN: any locked equilateral 3D chain? [Biedl et al.]
equilaterial 3D chain self-weaving on line [E. Demaine]
equilaterial unknotted closed chains? [M. Demaine]
equilaterial trees? [E. Demaine; Poon]
equililateral chain of equal-width cylinders? [O'Rourke]