2 meanings of "folding": (origami)
- folded state = description of paper after folding
- folding motion = continuum of folded states

we've focused on states, but in reality want motion

Equivalence: [Demaine, Devadoss, Mitchell, O'Rourke 2004]
any simple polygonal piece of paper has a
folding motion into any desired folded state

Proof: (1D)

(2D)

OPEN: what if paper has holes?
unknotted polyhedral paper? (for flattening)
OPEN: do finite number of extra creases suffice, if target folded state does not touch itself?
- above, all points become crease points

Rigid origami: what folds without extra creases?
- faces of crease pattern = rigid polygons
- creases = hinges

Example: [Balkcom, Demaine, Demaine, Ochsendorf, You 2006]
- paper shopping bag doesn't fold rigidly
  (for \( h > \frac{w}{2} \), standard crease pattern)
- 2 folded states: open & flat
- no folding motions

Little known about rigid origami; looking for good open questions
**Linkages:**

Graph = vertices (V) & edges (E) (connectivity/combinatorial structure)

Linkage = graph + lengths of edges (l: E \rightarrow \mathbb{R}^+) (intrinsic geometry)

[+ coordinates for pinned vertices (p: V' \rightarrow \mathbb{R}^d)]

Configuration of a linkage in \mathbb{R}^d

= coordinates for vertices (C: V \rightarrow \mathbb{R}^d) satisfying constraints of linkage

\( ||C(v) - C(w)|| = l(v,w) \) for all \( \forall v, w \in E \)

\( C(v) = p(v) \) for all \( v \in V' \)

(allowing intersections for this lecture)

Example:

![Graphs and linkages with configurations](image)

Motion (of a linkage in \mathbb{R}^d) = continuum of configurations (m: [0,1] \rightarrow C)
Configuration space = all configurations of a linkage
- view configuration of n-vertex linkage in $\mathbb{R}^d$
as (special) point in $\mathbb{R}^{dn}$:
  $$C=(\ldots,\ldots,\ldots;\ldots,\ldots,\ldots;\ldots,\ldots,\ldots)$$
  \(d\) coords for $v_1$  \(v_2\)  \(d\) coords for $v_n$

$\Rightarrow$ configuration space = subspace of $\mathbb{R}^{dn}$
- motion = path/curve in configuration space
- square example: $n=4, d=2$
  $\Rightarrow$ configuration space lives in $\mathbb{R}^8$
  - 4 dimensions fixed by pinning
  - locally one dimensional; topologically:

Degrees of freedom = local intrinsic dimension of configuration space around configuration
- intuitively: $d \cdot (#\text{unpinned vertices}) - (#\text{edges})$
  (but in reality, some edges are extraneous - see L3)

Trajectory of a vertex in a linkage = all points that vertex can reach in configurations
  ( = projection of configuration space onto vertex’s coords)
Kempe’s Universality Theorem: [Kempe 1876 had bug; Thurston; King 1999; Kapovich & Millson 2002; Abbott, Barton, Demaine 2008]
Any algebraic planar curve \( \Phi(x,y) = \sum \Sigma_i c_i x^p_i y^q_i = 0 \), intersected with any bounded disk, is exactly the trajectory of a vertex of some linkage.

Kempe’s “proof”:
- start with rhombus to constrain point \( p \) within disk:

- goal: constrain \( p=(x,y) \) to satisfy \( \Phi(x,y)=0 \)

Main trick: use trig. to effectively “take logarithm”
- \( x = \frac{r}{2} \cos \alpha + \frac{r}{2} \cos \beta \)
- \( y = \frac{r}{2} \sin \alpha + \frac{r}{2} \sin \beta = \frac{r}{2} \cos (\alpha - \frac{\pi}{2}) + \frac{r}{2} \cos (\beta - \frac{\pi}{2}) \)
- apply trig. identity
  \[ \cos A \cos B = \frac{1}{2} [\cos (A+B) + \cos (A-B)] \]
  to polynomial \( \Phi(x,y) = \sum \Sigma_i c_i x^p_i y^q_i \)
  \[ \Rightarrow \Phi(x,y) = c + \sum \Sigma_i c_i \cos \left( \frac{r}{2} \alpha + \frac{r}{2} \beta + \delta_i \right) \]
  \( \gamma \) or \( \pm \pi/2 \)
Kempe’s “proof” (cont’d)

- new goal: construct line segment of length $c_i$ & angle $r_i \alpha + s_i \beta + \delta_i$, for each $i$

- force final vertex on line $x = -c$ via large Peaucellier linkage

- build “machine” for angle arithmetic with ops:
  - multiply given angle by integer
  - add two given angles
  - copy an angle from one place to another
Kempe’s gadgets:

**Contraparallelogram:**
- opposite sides equal & self-crossing (not parallelogram)
  ⇒ opposite angles equal; α determines β

**Multiplier:**
- k similar contraparallelograms sharing their β’s ⇒ equal α’s
- can be more efficient — (O(\log k)) edges — by repeated doubling, but this will not affect final complexity

**Additor:**
- use 2 times multipliers to
  - bisect angle between segments
  - reflect x axis through bisector

**Translator:** two parallelograms
- opposite edges parallel & same length
- make adjacent edges long (& same) for reach
- could use big rhombus — but this construction allows arbitrary length of input (or output) edge
Bug: [Kapovich & Millson]
- parallelograms can flip to contraparallelograms & vice versa via degenerate (flat) configuration
  ⇒ Kempe proved weaker result: trajectory includes desired poly curve & more
- fix for parallelogram:

- different, messier construction for complex polynomials
- fix for contraparallelogram: [Abbott & Barton 2004]

Application:
Sign name via Weierstrass approximation theorem:
any continuous function $f: [a, b] \rightarrow \mathbb{R}$ has an $\epsilon$-approximate polynomial $p - |p(x) - f(x)| \leq \epsilon$ for all $x \in [a, b]$ for any $\epsilon > 0$ (apply to each coordinate of curve)
Generalizations/strengthenings:
- curves/surfaces in $d$ dimensions
- $\Theta(n^d)$ bars is optimal for degree $n$
- any compact semialgebraic set (d-dim) (bounded system of polynomial $\leq$ inequalities)
as vertex trajectory
- coNP-hard to test rigidity
- configuration space = union of finitely many analytically isomorphic copies of any desired algebraic set (any # dim.) mapping & inverse have local power-series expansion

[Abbott, Barton, Demaine 2008]

[Open] What if edges are forbidden from crossing? [Shimamoto 2004]

[Project]: implement Kempe applet

[Project]: sculpture based on Kempe linkage/gadgets

[Project]: design linkages for letters of alphabet (e.g. letter C: http://www.jimloy.com/cindy/cindy.htm)

Application: constructing algebraic numbers in origami via alignments [GFALOP 19.5; cf. Alperin & Lang 2006]