Universal hinge patterns: (for origami transformers)
[Benbernou, Demaine, Demaine, Ovadya 2010]
- Suppose crease pattern required to be subset of fixed "hinge pattern"
  (e.g. Origamizer uses completely different creases for every model)
- n x n box-pleat pattern can make any polycube of $O(n)$ cubes, seamless:
  - Cube gadget turns $O(1)$ rows & columns into a cube sticking out of sheet
  - Even if bumps elsewhere (not in eaten rows/cols)

- To make a tree of cubes: (= any polycube)
  - Make a leaf
  - Conceptually remove it
  - Repeat
- Actually need to reserve space ahead of time for all the cube gadgets
- $\Theta(n)$ cubes is optimal in worst case:  1x1xn needs diameter $\Omega(n)$
- but sometimes can do better:

Maze folding: [Demaine, Demaine, Ku 2010]
any $n \times n$ orthogonal maze extruded from square can be folded from $\Theta(n) \times \Theta(n)$ square
- constant scale factor! (3 for unit extrusion)
- gadget for each possible vertex:

  . 1 2 2 3 4

  degree 0 1 2 3 4

- designed to have compatible interfaces:
  - ridge for maze edges
  - flat "double pleat" for nonedges
- cut & paste

try it out: http://erikdemaine.org/maze/
Origami design is hard ~ how to formalize?

NP-hard ≈ “computationally intractable”
- if a problem is NP-hard, then there’s no efficient algorithm to solve it unless $P=NP$
  (famous unsolved problem, worth $1 M +$)
- $P \neq NP$ ≈ “computers can’t simulate lucky guessing, say heads vs. tails, without trying both options”
  ≈ almost everyone believes it

Examples of NP-hard problems:
- **Partition**: given n integers, can you split them into two halves of equal sum?
  (e.g. equalizing teams for a game)
  - actually only hard for exp. large integers: “weakly NP-hard”
- **SAT**: given Boolean formula $(x \text{ AND NOT } y) \text{ OR } z$
  can you set the variables $x \sim y \sim z$
  true/false so that formula is true?

Approach: show e.g. Partition is easier / a special case of your problem: any Partition problem can be converted into a problem of your type
  $\Rightarrow$ your problem is NP-hard too
Simple example: (from Problem Session 1)

given single-vertex hinge pattern,
is some subset of \( \{\Theta\} \) creases flat foldable? (posed by student after class)

is NP-hard:
- given Partition problem, scale integers uniformly so that their sum = 360°
- angles of single-vertex crease pattern
  looking at angular travel (Kawasaki-Justin),
at each hinge can crease \( \Rightarrow \) change direction
  or not \( \Rightarrow \) same direction

\( \Rightarrow \) can choose \( +\Theta_i \) or \( -\Theta_i \) for each \( i \)
- must have \( \sum_i \pm \Theta_i = \emptyset \)
  i.e. \( \sum_i \Theta_i = \sum_i -\Theta_i \)
- yes to Partition \( \iff \) yes to flat-foldable CP
Simple folds: can given crease pattern be folded flat by sequence of simple folds?
- saw how to solve for 1D patterns & 2D orthogonal maps:

NP-hard if we add 45° diagonal creases or allow orthogonal paper

[Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena 2002]
- reduction from Partition (weakly NP-complete)

- fold some of horiz. creases
- fold a vertical crease
- if hit wall: stuck
- else can fold other vertical & remaining horiz. creases
Global flat foldability:  
① deciding flat foldability of given crease pattern is strongly NP-hard  
② constructing valid layer ordering for given flat-foldable mountain-valley pattern is strongly NP-hard

Proof: (①) reduce from all-positive not-all-equal 3-satisfiability: given triples \( (x_i, x_j, x_k) \), is there a Boolean assignment to \( x_1, x_2, \ldots, x_n \) such that no triple is all-true or all-false? (strongly NP-hard, like SAT)

\[ \text{Wire} = \text{"pleat"} = \text{two close parallel creases} \]
false \( \iff \) left mountain \( \rightarrow \) \( \rightarrow \) \( \rightarrow \) false \( \rightarrow \) true

\[ \text{NAE clause} = \text{triangular "overtwist"} \]
- can’t all fold same way  
- (twist is borderline)

Reflector splits wire \( x \) into two copies, one negated  
\( \Rightarrow \) split gadget \( \Uparrow \) \( \Uparrow \)  
& turn gadget \( \Uparrow \) (with noise)  
\( \Rightarrow \) can connect variable wires to desired clauses  

Also need crossover gadgets.

[Bern & Hayes 1996]
Disk packing: [Demaine, Fekete, Lang 2010] can you place \( n \) given disks nonoverlapping with centers in given square? is NP-hard

- reduction from 3-Partition: given \( n \) integers, can you split them into \( n/3 \) triples of equal sum? strongly NP-hard \( \sim \) integers \( = O(n) \), not exp.

- lots of disks to force identical pockets & make all other pockets too small

- within one pocket:

  \[
  \begin{align*}
  &a_i \text{ oversized} \\
  &\text{undersized by desired triple sum} \\
  &a_j \text{ oversized} \\
  &a_k \text{ oversized}
  \end{align*}
  \]

\( \Rightarrow \) just fits if 3-partition