Single-vertex crease pattern (without loss of generality)

= disk of paper, creases emanate from center
  Idea: capture local foldability around a vertex
  = circular sequence of angles $\theta_1, \theta_2, \ldots, \theta_n$
    normally, $\theta_1 + \theta_2 + \ldots + \theta_n = 360^\circ$
    allow other sums, especially $\leq 360^\circ$ (convex cone)
    in particular for induction

Flat folding = folding of 1D circle (boundary of disk) on the circle
  = folding of 1D circle onto line
  (assuming convex cone & at least one fold
  $\Rightarrow$ can't reach all the way around circle)

Differences from 1D (segment) flat folding:
- not all crease patterns flat foldable:
- alternating M/V can fail:
- equilateral $\Rightarrow$ all mountain-valley patterns possible
  e.g. all valleys
- mingling $\Rightarrow$ existence of crimp:
  could have \ldots (][[]][]\ldots circularly
Characterization of flat-foldable single-vertex crease pat:

[Kawasaki 1989; Justin 1989; Hull 1994]

\( \Theta_1, \Theta_2, \ldots, \Theta_n \) is flat-foldable convex cone

\[ \iff \Theta_1 + \Theta_3 + \ldots + \Theta_{n-1} = \Theta_2 + \Theta_4 + \ldots + \Theta_n \quad \text{\& \ n \ even} \]

\[ = 180^\circ \text{ for flat paper} \]

**Proof:**

(\( \Rightarrow \)) angles \( \Theta_i \) measure travel on circle/line
- creases switch direction of travel

\[ \Rightarrow n \text{ must be even (cycle of switches)} \]
- total motion = \( \pm (\Theta_1 - \Theta_2 + \Theta_3 - \Theta_4 + \ldots + \Theta_{n-1} - \Theta_n) \)
  - total motion = 0 to end where we started (assuming convex cone \& at least one fold—
  - else = 0 (mod 360°)

\[ \Rightarrow \text{alternating sum of angles } = 0 \]

(\( \Leftarrow \)) cut at an "extreme" crease (e.g., leftmost)
- 1D segment crease pattern
- fold flat e.g. accordion
- two ends corresponding to cut crease are aligned because total motion = 0
  & can join because extreme

**Nonconvex cone:** \( \Theta_1 - \Theta_2 + \ldots = 0 \text{ or } \pm 360^\circ \) [Demaine & O'Rourke 2007]
Flat-foldable single-vertex mountain-valley patterns

Count: \( \# \text{mountains} - \# \text{valleys} = \pm 2 \)

in convex cone

Proof: measure total turn angle = 180° - interior angle

(>0 for convex, <0 for reflex vertices)
- mountain turns +180°, valley turns -180°
- small turn caused by circle, but cancels out assuming convex cone \( \Rightarrow \) can’t reach around
- no crossing \( \Rightarrow \) total turn angle = \( \pm 360° \)
  \[ 180° \cdot \# \text{mountains} - 180° \cdot \# \text{valleys} = \pm 360° \]
  \[ \Rightarrow \# \text{mountains} - \# \text{valleys} = \pm 2. \]

Nonconvex cones: if \( \Theta_1 - \Theta_2 + \cdots = \pm 360° \), \#M - \#V = 0

Generic case: strict local minimum angle is surrounded by one mountain & one valley

\[ \downarrow \quad \rightarrow \quad \downarrow \]

\[ \Rightarrow \text{can immediately crimp any such angle} \]
- preserves flat foldability as before for 10 segments:

Remaining case: equal angles
Characterization of flat-foldable single-vertex mountain-valley pattern:

[Hull 2001 & 2003; Demaine & O’Rourke 2007]

**Local counts:** Among $k$ equal angles surrounded by strictly larger angles (e.g., globally smallest angle),

- $\#$ mountains $- \#$ valleys $=\begin{cases} 0 & \text{if } k \text{ is odd} \\ \pm 1 & \text{if } k \text{ is even} \end{cases}$

**Proof:** build cone from $k$ equal angles & larger angles

- if $k$ even then extend one larger angle to match the other

- if $k$ odd then add new angle of $\leq$ larger angles $-$ equal angle

$\implies$ flat folding of cone with same M-V assign. & $\Theta_1 - \Theta_2 + \cdots + \Theta_{n-1} + \Theta_n = 0$

- Maekawa’s Theorem $\implies$ $\#$ mountains $- \#$ valleys $= \pm 2$

(cone might be nonconvex but still $\Theta_1 - \Theta_2 + \cdots = 0$)

- if $k$ even then one new crease

$\implies$ $180^\circ$ turn + new crease $\implies$ $\#M - \#V = \pm 1$

- if $k$ odd then two new creases (same dir.)

$\implies$ $0^\circ$ turn + 2 new creases $\implies$ $\#M - \#V = 0$, \(\square\)

$\implies$ there is at least one crimp among these creases

- applies unless all angles are equal

$\implies$ crimp exists by $\#$ mountains $- \#$ valleys $= \pm 2$ (or 0)

- unless just 2 creases $\implies$ same direction (or opposite direction if $\Theta_1 - \Theta_2 = \pm 360^\circ$)

- linear-time algorithm (maintain crimps)
Combinatorics of single-vertex mountain-valley patterns:

- Smallest in generic case $\Rightarrow 2^{n/2}$ choices per crimp
- Largest in equal-angle case $\Rightarrow 2^{\frac{n}{\sqrt{2}} \cdot \left(\frac{n}{\sqrt{2}} - 1\right)}$
  $\#M - \#V = 2$ or $-2$ --- Ms & Vs

OPEN: polynomial-time characterization for $k$-vertex crease pattern, $k$ small?

- $n^{f(k)}$? \quad $f(k)$ \quad $n^{o(1)}$?

(We will see in Lecture 5 that the general problem is NP-hard)
Local foldability: [Bern & Hayes 1996]

linear-time algorithm finds a consistent mountain-valley assignment (if possible)
Such that each vertex locally folds flat

Proof: all possible mountain-valley assignments of a single vertex generated by crimps
- crimped pair forced unequal
- final pair forced equal
- cycles can have parity issue:
  - pairing unique in generic case
  - if equal angle next to crimped angle then can interchange order of crimps
  - merge if interchange decreases \# paths/cycles of =/≠ constraints
  - merges only fix parity problems (lemma)
  □

**PROJECT**: implement local foldability test
converting crease pattern → M-V pattern

**OPEN**: minimum number of added creases to make given crease pattern [locally] flat foldable
- with or without mountain-valley assignment
- always possible via disk-packing fold & cut
Tree method: [Lang 1994–2003; Lang, Demaine, Demaine 2004–] algorithm to find folding of smallest square into “uniaxial” origami base whose projection is a desired metric tree

Output

- non-self-intersection is only conjectured
  (we’re working on it)

More next lecture!