Definitions and Notation

Problem 1: Define the following words, phrases and symbols.
1. Set $A = \{x, y\}$, subset $B \subseteq A$, proper subset $B \subset A$, multiset $\{x, y, y\}$, power set $P(A)$, cardinality $|A|$, infinite set, natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$, integers $\mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}$, empty set $\emptyset$, union $A \cup B$, intersection $A \cap B$, Cartesian product $A \times B$, complement $\overline{A}$, sequence $(x, y)$, $k$-tuple $(x_1, x_2, \ldots, x_k)$.
2. Function $f : D \rightarrow R$, domain $D$, range $R$, mapping $\rightarrow$, one-to-one, onto, bijection (one-to-one, onto).
3. Relation $R = \{(d_1, r_1), (d_2, r_2), \ldots, (d_i, r_i)\}$, reflexive $\forall x, xRx$, symmetric $\forall x, y, xRy \iff yRx$, transitive $\forall x, y, z, xRy \land yRz \Rightarrow xRz$, equivalence (reflexive, symmetric, transitive).
4. Graph $G = (V, E)$, degree, path, simple path, cycle, strongly connected.
5. Alphabet (input/output) $\Sigma = \{a, b, c\}$, symbols $a$, string $w = baac$, length $|w|$, empty string $\epsilon$, substring (consecutive) $baa$, concatenation $w || w$ or $uw$, lexicographic ordering $(e, 0, 1, 00, 01, 10, 11, 000, \ldots)^2$, language $L = \{w_1, w_2, \ldots, w_k\}$.
6. Boolean logic $\{0, 1\}$, NOT $\neg p$, AND $p \land q$, OR $p \lor q$, XOR $p \oplus q$, implication $p \Rightarrow q$, equality $p \iff q$, distributive law $p \land (q \lor r) \iff (p \land q) \lor (p \land r)$.
7. Theorem, lemma, corollary, proof, intuition, induction (assumes $P(n)$), strong induction (assumes $P(0), P(1), \ldots, P(n)$).
8. (*) Machine, string accepted by a machine, language recognized by a machine.

Proof Techniques

Problem 2: Set-Theoretic Equivalence: Recall that in order to prove two sets $A, B$ are equivalent, one must show that $A \subseteq B$ and $B \subseteq A$. Prove De Morgan’s Law that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Problem 3: Proof by Contradiction: If there are 6 people at a party shaking hands, then there must be at least two people who shook hands with the same number of other people.

Problem 4: Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step): Problem 0.11 from Sipser’s Text. Find the error in the following proof that all horses are the same color.
Claim: In any set of $h$ horses, all horses in the set are the same color.
Proof: By induction on $h$.
Basis: For $h = 1$. In any set containing just one horse, all horses clearly are the same color.
Inductive Step: For $k \geq 1$, assume that the claim is true for $h = k$ and prove that it is true for $h = k + 1$. 

1Sipser, pg 4. Zero can also be included in $\mathbb{N}$.
2Observe the anomaly that 11 preceeds 000; length takes precedence.
Take any set $H$ of $k + 1$ horses. We will show that all horses in this set are the same color. Remove one horse from this set to obtain the set $H_1$ with just $k$ horses. By the induction hypothesis, all the horses in $H_1$ are the same color. Now replace the removed horse and remove a different one to obtain the set $H_2$. By the same argument, all the horses in $H_2$ are the same color. Therefore, all the horses in $H$ must be the same color and the proof is complete.

**Problem 5: Proof by Induction (Base Case) (Induction Hypothesis) (Inductive Step):** Now correctly prove the following statement: $\forall n \in \mathbb{N}, n^3 - n$ is divisible by 6.

**Problem 6: Proof by Contradiction:** Give a proof by contradiction of the statement of Problem 5. (Start by assuming that for some $n \in \mathbb{N}$, $n^3 - n$ is not divisible by 6).

**Problem 7: Double Induction:**

Let the function $R(s, t)$ (for $s, t \in \mathbb{N}$) be defined by the induction:

$$R(s, t) = R(s, t - 1) + R(s - 1, t)$$

and the base cases

$$R(s, 2) = R(2, s) = s \quad (\text{for all } s \in \mathbb{N})$$

Prove that $R(s, t) \leq \binom{s + t - 2}{s - 1}$. 

1: Math Review-2