
Problem 1: (Sipser 7.20) Let $G$ represent an undirected graph and let

$$\text{SPATH} = \{ (G, a, b, k) \mid G \text{ contains a simple path of length at most } k \text{ from } a \text{ to } b \}$$

and

$$\text{LPATH} = \{ (G, a, b, k) \mid G \text{ contains a simple path of length at least } k \text{ from } a \text{ to } b \}$$

1. Show that $\text{SPATH} \in \mathbb{P}$.
2. Show that $\text{LPATH}$ is NP-complete. You may assume the NP-completeness of $\text{UHAMPATH}$, the Hamiltonian path problem for undirected graphs.

Problem 2: (Sipser 7.) For a cnf-formula $\phi$ with $m$ variables and $c$ clauses, show that you can construct in polynomial time an NFA with $O(cm)$ states that accepts all non-satisfying assignments, represented as Boolean strings of length $m$. Conclude that the problem of minimizing NFAs cannot be done in polynomial time unless $P = NP$.

Problem 3: An edge-cover in a graph $G(V, E)$ is a set of edges $E' \subseteq E$ of $G$ such that each vertex in $G$ is the end-point of at least one of the edges in $E'$. As a language,

$$\text{EDGE-COVER} = \{ (G, k) \mid G \text{ is an undirected graph that has an edge-cover with at most } k \text{ edges} \}$$

Show that $\text{EDGE-COVER} \in \mathbb{P}$. (Recall the problem $\text{VERTEX-COVER}$ that we proved NP-complete in the class.)

Problem 4: Suppose there exists a family of bijections $\{f_k\}_{k=1}^{\infty}$ such that $f_k$ maps integers of length $k$ onto integers of length $k$. We also know that

- For all $k$, $f_k$ is computable in polynomial time (in $k$), and
- No $f_k^{-1}$ is computable in polynomial time.

Prove that this would imply that the language

$$A = \{ \langle x, y \rangle \mid f^{-1}(x) < y \}$$

is in $(\text{NP} \cap \text{coNP}) \setminus \mathbb{P}$. 