This problem set contains some harder-than-usual problems. In solving them, you can call upon everything you have learned so far about finite-state automata and regular languages.

**Problem 1: Distinguishable strings and indices** (From Sipser Problems 1.51 and 1.52)

Let \( x \) and \( y \) be strings and let \( L \) be any language (not necessarily regular). We say that \( x \) and \( y \) are distinguishable by \( L \) if some string \( z \) exists such that exactly one of the strings \( xz \) and \( yz \) is in \( L \). On the other hand, if for all strings \( z, xz \) is in \( L \) if and only if \( yz \) is in \( L \), we say that \( x \) and \( y \) are indistinguishable by \( L \). If \( x \) and \( y \) are indistinguishable by \( L \), we write \( x \equiv_L y \).

(a) Show that \( \equiv_L \) is an equivalence relation.

Now let \( L \) be a language and \( X \) a set of strings. We say that \( X \) is pairwise distinguishable by \( L \) if every two distinct strings in \( X \) are distinguishable by \( L \). Define the index of \( L \) to be the maximum number of elements in any set that is pairwise distinguishable by \( L \). In other words, the index of \( L \) is equal to the number of equivalence classes in \( L \), which may be finite or infinite.

(b) Let \( L_1 \) be the regular language \((01)^*00\). What is the index of \( L_1 \)? Describe the equivalence classes.

(c) Build a DFA for \( L_1 \) with states corresponding to the equivalence classes (i.e., the number is states is equal to the index of \( L_1 \)).

(d) Let \( L_2 \) be the non-regular language \( \{0^n1^n : n \geq 1\} \). What is the index of \( L_2 \)? Describe the equivalence classes.

(e) Now consider an arbitrary language \( L \). Prove that if \( L \) is recognized by a DFA with \( k \) states, then \( L \) has index at most \( k \).

(f) Again consider an arbitrary language \( L \). For \( L \) with index \( k \), describe how to construct a DFA with \( k \) states.

We can conclude from this problem that a language \( L \) is regular if and only if it has a finite index. Moreover, its index is the size of the smallest DFA recognizing it.

**Problem 2: Inequivalent DFAs** Suppose that two DFAs \( M_1 = (Q_1, \{0, 1\}, \delta_1, q_0, F_1) \) and \( M_2 = (Q_2, \{0, 1\}, \delta_2, q_0, F_2) \) over alphabet \( \{0, 1\} \) recognize different languages.

(a) In terms of the sizes of the state sets \( Q_1 \) and \( Q_2 \), determine an upper bound \( u \) on the length of the smallest string on which machines \( M_1 \) and \( M_2 \) must give different answers (accept vs. reject). That is, determine some \( u \) such that \( M_1 \) and \( M_2 \) must actually give different results for some string of length \( \leq u \).

(b) Prove your answer to (a).