Transfer Functions and Z Transforms

- Basic idea of Z-transform
- Transfer functions represented as ratios of polynomials
- Composition of functions is multiplication of polynomials
- Black's formula

Difference Equations

Suppose we have a system, like a discrete-time robot controller, that can be described using a difference equation like

\[ y[n] = 2y[n-1] + x[n] \]

where \( x \) is the sequence of input values and \( y \) is the sequence of output values. This, along with initial conditions, is a complete characterization of how the system behaves. But it isn't very easy for us to look at it and understand what's going to happen, and it's relatively inefficient \( O(n) \) to compute the \( n \)th output value. So, we'd like to find another expression for \( y[n] \) that's easier to work with.

We can't do this in the general case, but if we restrict the form of \( x[n] \) to be \( Xa^n \), where \( X \) and \( a \) are constants, then everything will work out nicely. Two useful cases of this are when \( a = 1 \), so the input is just constant over time, and when \( a = e^{i\omega} \), so the input is a sinusoid.

We can divide up our characterization of the behavior into three questions:

1. Is this system going to converge to a constant or stable behavior?
2. If so, what will it converge to?
3. How will it behave on its way to convergence?

Question 1 can be answered without reference to the input or the initial conditions; question 2 depends on the input, and question 3 depends on both the input and the initial conditions.

For linear difference equations, all solutions will have the form

\[ y[n] = h_i[n] + p_x \]

where \( p_x \) is a particular solution, describing what the system will eventually converge to for a given input \( x \), and \( h_i[n] \) is a homogeneous solution, which describes the behavior of the system with an input of zero.

The general method for solving such equations is:

1. Find the general form of the homogeneous solution, which will be a sum of terms \( \lambda_j^n \) where the \( \lambda_j \) are natural frequencies (i.e., the roots of the characteristic polynomial). Whether or not the system converges depends only on the magnitudes of the natural frequencies.

2. Find a particular solution for the given input \( x \).
3. Pick the homogeneous solution that, when added to the particular solution, matches the given initial conditions.

We'll see how to do this in detail by working through two examples.

First-Order Example

Let's start with the following difference equation:

$$y[n] = \frac{1}{2(k+1)}y[n-1] + \frac{k}{k+1}x[n] .$$

This is called a first-order difference equation, because it only depends on the value at one previous time step. We can think of $k$ as a gain in the system; we'll leave it as a variable, so we can think of what values for $k$ will result in which kinds of behavior of the system.

Will it converge? The first job is find the roots of the characteristic polynomial (same as the poles of the transfer function). If we got to this difference equation from a transfer function, then the poles are the roots of the polynomial in the denominator. But if someone just hands us a difference equation, we can find the characteristic polynomial by ignoring the input term, and assuming that $y[n] = z^n$ for some unknown $z$. In that case, we'll get

$$z^n = \frac{1}{2(k+1)}z^{n-1} ,$$

which we can reduce, by dividing through by $z^{n-1}$ to

$$z = \frac{1}{2(k+1)} .$$

Generally, we'll have to solve this for $z$, but in this case we're already done, and so we know that there is a single root at $1/2(k+1)$.

Here's what the poles (roots) tell us about the convergence of the system:

- If all poles have magnitude $< 1$, and there is no input, then the system converges to 0; if there is input it will converge to something.
- If all poles are real and positive, then the system will converge monotonically (no oscillation).
- Complex or negative poles will cause oscillation; frequency of the oscillation is a function of the angle of the pole in the complex plane.

In our case, the pole is real. The system will converge if $|\frac{1}{2(k+1)}| < 1$, which will happen if $k > -\frac{1}{2}$ or $k < -\frac{3}{2}$ (these two-sided inequalities are tricky; it's easy to forget both sides or get them wrong). In the first case, it will converge monotonically; in the second case, it will oscillate.
What will it converge to? Now it’s time to find the particular solution. This is the “steady state” behavior of the system. It is well-defined if the system converges, but not otherwise. It is a constant, if the input is constant; a sinusoid if the input is a sinusoid. It is a description of what the system will do when the transient behavior (described by the homogeneous solution) dies out.

The steady-state behavior depends on the input. (That makes sense, right? It would be hard to say what the output will be, without knowing what the input is.) Our approach to this problem will only work if the input can be written as $x[n] = Xa^n$. If that’s true, and $y$ is defined as a linear difference equation with input $x$, then $y[n]$ is guaranteed to have the form $Ya^n$.

So now we can rewrite our difference equation as

$$Ya^n = \frac{1}{2(k+1)} Ya^{n-1} + \frac{k}{k+1} Xa^n.$$ 

Remember that this form only captures the steady-state behavior. In this example, we’ll assume that $x[n] = 1$ for all $n$, which means that $X = 1$ and $a = 1$. Thus, our equation will simplify to

$$Y = \frac{1}{2(k+1)} Y + \frac{k}{k+1}.$$ 

Now we’re looking for a value of $Y$ that is a “fixed point” of this equation; that is, if $y[n]$ ever has this value, then it will continue to have this value forever into the future. We can do this by solving for $Y$, obtaining

$$Y = \frac{2k}{2k+1}.$$ 

That is the particular solution to the problem. Note that it depends on the value of the gain.

How will it behave on the way to convergence? Now we know when the system will converge and, if it does, what value it will converge to. What will it do along the way? To figure this out, we need to write out the whole form of the solution, plug in the things we know, and solve for what we don’t, using the initial conditions.

The general form of the total solution is

$$y[n] = h_i[n] + px ,$$

which can be expanded as

$$y[n] = \sum_{j=1}^{m} A_j \lambda_j^n + Ya^n ,$$

where $m$ is the number of roots, the $\lambda_j$ are the roots, the $A_j$ are constant coefficients, and $Ya^n$ is the particular solution. The homogeneous solution describes the system’s behavior with zero input; you can verify that it has this form by plugging it into the original equation (with $x[n] = 0$).

We can see that if the roots have magnitude $< 1$, then they’ll decay away as $n$ grows, and we’ll be left with the particular solution. We have one root, and we know the particular solution; so all that’s missing is a constant:

$$y[n] = A \left( \frac{1}{2(k+1)} \right)^n + \frac{2k}{2k+1} .$$
Now, we use our knowledge of the initial conditions. Let’s assume that \( y[n] = 0 \) for all \( n < 0 \). So
\[
y[-1] = A \left( \frac{1}{2(k+1)} \right)^{-1} + \frac{2k}{2k+1} = 0.
\]
Solving this for \( A \), we get
\[
A = -\frac{k}{(k+1)(2k+1)}.
\]
So, our final result is
\[
y[n] = -\frac{k}{(k+1)(2k+1)} \left( \frac{1}{2(k+1)} \right)^n + \frac{2k}{2k+1}.
\]
Yay. This has exactly the same value at the integer arguments as the original difference equation and it can be computed efficiently. We also, in the process, understood a lot about how the system behaves, and gained intuition about how to set \( k \) to get a desired behavior.

**Second-Order example**

Okay. Let’s do one more example, this time with a second-order system:
\[
y[n] = -\frac{1}{k}y[n-1] + \frac{1}{2k}y[n-2] + x[n].
\]
This is called a second-order difference equation, because it only depends on the value at two previous steps.

Will it converge? This time, the characteristic polynomial is
\[
z^n = -\frac{1}{k}z^{n-1} + \frac{1}{2k}z^{n-2},
\]
which we can reduce, by dividing through by \( z^{n-2} \) to
\[
z^2 = -\frac{1}{k}z + \frac{1}{2k}.
\]
Solving this for \( z \), we get two roots:
\[
\lambda_1 = \frac{-1 + \sqrt{1+2k}}{2k},
\]
\[
\lambda_2 = \frac{-1 - \sqrt{1+2k}}{2k}
\]
It’s clear that these roots are only real when \( k > -1/2 \). The system will converge only if both roots have magnitude < 1. When real, the first root, \( \lambda_1 \), is < 1 when \( k > -1/2 \). The second root is < 1 when \( k > 3/2 \). The system is only stable when both roots have magnitude < 1. So the system is stable and monotonically convergent when \( k > 3/2 \). When the roots are complex, they both have magnitude < 1 when \( k < -1/2 \). (Remember that the magnitude of a complex number \( a + bj \) is \( \sqrt{a^2 + b^2} \).) So, the system is stable, but oscillates toward convergence, when \( k < -1/2 \).
What will it converge to? To capture the steady-state behavior, we can rewrite our difference equation as

\[ Y a^n = -\frac{1}{k} Y a^{n-1} + \frac{1}{2k} Y a^{n-2} + X a^n. \]

Remember that this form only captures the steady-state behavior. In this example, we’ll assume that \( x[n] = 1 \) for all \( n \), which means that \( X = 1 \) and \( a = 1 \). Thus, our equation will simplify to

\[ Y = -\frac{1}{k} Y + \frac{1}{2k} Y + 1. \]

Solving for \( Y \), we get a particular solution of

\[ Y = \frac{2k}{2k+1}. \]

How will it behave on the way to convergence? We’ll start with the general form again:

\[ y[n] = \sum_{j=1}^{m} A_j \lambda_j^n + Y a^n, \]

and then plug in what we know, to get

\[ y[n] = A_1 \left( \frac{-1 + \sqrt{1+2k}}{2k} \right)^n + A_2 \left( \frac{-1 - \sqrt{1+2k}}{2k} \right)^n + \frac{2k}{2k+1}. \]

It’s more complicated this time, because we have two roots.

Now, we use our knowledge of the initial conditions. Let’s assume that \( y[n] = 0 \) for all \( n < 0 \). Because we have to solve for both constants \( A_1 \) and \( A_2 \), we need to use two previous values:

\[
\begin{align*}
y[-1] &= A_1 \left( \frac{-1 + \sqrt{1+2k}}{2k} \right)^{-1} + A_2 \left( \frac{-1 - \sqrt{1+2k}}{2k} \right)^{-1} + \frac{2k}{2k+1} \\
y[-2] &= A_1 \left( \frac{-1 + \sqrt{1+2k}}{2k} \right)^{-2} + A_2 \left( \frac{-1 - \sqrt{1+2k}}{2k} \right)^{-2} + \frac{2k}{2k+1}
\end{align*}
\]

This is kind of hairy, but still quadratic, so we can solve to get

\[
\begin{align*}
A_1 &= \frac{k}{(1+2k)(1+\sqrt{1+2k})} \\
A_2 &= \frac{-1 - \sqrt{1+2k}}{2(1+2k)}
\end{align*}
\]

So, our final result is

\[ y[n] = \frac{k}{(1+2k)(1+\sqrt{1+2k})} \left( \frac{-1 + \sqrt{1+2k}}{2k} \right)^n + \frac{-1 - \sqrt{1+2k}}{2(1+2k)} \left( \frac{-1 - \sqrt{1+2k}}{2k} \right)^n + \frac{2k}{2k+1}. \]