Lecture 18: Shortest Paths IV - Speeding up Dijkstra

Lecture Overview
- Single-source single-target Dijkstra
- Bidirectional search
- Goal directed search - potentials and landmarks

Readings
Wagner paper on website (upto Section 3.2)

DIJKSTRA single-source, single-target

Initialize()
\[ Q \leftarrow V[G] \]
while \( Q \neq \emptyset \)
do \( u \leftarrow \text{EXTRACT\_MIN}(Q) \) (stop if \( u = t \! \))
for each vertex \( v \in \text{Adj}[u] \)
do \( \text{RELAX}(u, v, w) \)

Observation: If only shortest path from \( s \) to \( t \) is required, stop when \( t \) is removed from \( Q \), i.e., when \( u = t \)
Bi-Directional Search

Note: Speedup techniques covered here do not change worst-case behavior, but reduce the number of visited vertices in practice.

Bi-D Search

Alternate forward search from $s$
backward search from $t$
(follow edges backward)
$d_f(u)$ distances for forward search
$d_b(u)$ distances for backward search

Algorithm terminates when some vertex $w$ has been processed, i.e., deleted from the queue of both searches, $Q_f$ and $Q_b$

Subtlety: After search terminates, find node $x$ with minimum value of $d_f(x) + d_b(x)$. $x$ may not be the vertex $w$ that caused termination as in example to the left!
Find shortest path from $s$ to $x$ using $\Pi_f$ and shortest path backwards from $t$ to $x$ using $\Pi_b$. 

Note: $x$ will have been deleted from either $Q_f$ or $Q_b$ or both.
Minimum value for $d_f(x) + d_b(x)$ over all vertices that have been processed in at least one search (see Figure 3):

$$d_f(u) + d_b(u) = 3 + 6 = 9$$
$$d_f(u') + d_b(u') = 6 + 3 = 9$$
$$d_f(w) + d_b(w) = 5 + 5 = 10$$
Goal-Directed Search or $A^*$

Modify edge weights with potential function over vertices.

$$\bar{w}(u,v) = w(u,v) - \lambda(u) + \lambda(v)$$

Search toward target as shown in Figure 4.

**Correctness**

$$\bar{w}(p) = w(p) - \lambda_t(s) + \lambda_t(t)$$

So shortest paths are maintained in modified graph with $\bar{w}$ weights (see Figure 5).

To apply Dijkstra, we need $\bar{w}(u,v) \geq 0$ for all $(u,v)$.

Choose potential function appropriately, to be feasible.

**Landmarks**

Small set of landmarks $LCV$. For all $u \in V, l \in L$, pre-compute $\delta(u,l)$.

Potential $\lambda_t^{(l)}(u) = \delta(u,l) - \delta(t,l)$ for each $l$.

CLAIM: $\lambda_t^{(l)}$ is feasible.

**Feasibility**

$$\bar{w}(u,v) = w(u,v) - \lambda_t^{(l)}(u) + \lambda_t^{(l)}(v)$$

$$= w(u,v) - \delta(u,l) + \delta(t,l) + \delta(v,l) - \delta(t,l)$$

$$= w(u,v) - \delta(u,l) + \delta(v,l) \geq 0 \text{ by the } \Delta \text{-inequality}$$

$$\lambda_t(u) = \max_{l \in L} \lambda_t^{(l)}(u) \text{ is also feasible}$$
Figure 3: Forward and Backward Search and Termination.
Figure 4: Targeted Search

Figure 5: Modifying Edge Weights.