Problem 1. (Reducing the directed max flow to the undirected max flow.) Consider a directed graph $G = (V, E)$ (with unit capacities), a source vertex $s$ and a sink vertex $t$. Now, let $\hat{G} = (V, \hat{E})$ be an undirected graph over the same vertex $V$ set as $G$. The edges set $\hat{E}$ of $\hat{G}$ is defined as follows: for each arc $e = (u, v) \in E$ in $G$, with $u$ being the tail and $v$ being the head of $e$, the graph $\hat{G}$ has edges $(s, v), (u, v)$, and $(u, t)$ added to $\hat{E}$.

(a) Let $F^*$ be the value of the maximum $s$-$t$ flow in $G$. Argue that the value $\hat{F}^*$ of the maximum $s$-$t$ flow in $\hat{G}$ is exactly $2F^* + |E|$.

Hint: Use the max-flow min-cut theorem.

(b) (Extra credit) Design a nearly-linear time procedure that given a maximum $s$-$t$ flow $\hat{f}^*$ in $\hat{G}$ returns a maximum $s$-$t$ flow in $G$.

Hint: You might need to use here some advanced data structure result.

Note: This construction extends to arbitrary capacities in a straightforward manner.

Problem 2. (Implementing the conjugate gradient method.) Recall the linear system solving via conjugate gradient method that we discussed in class.

Algorithm 1 Conjugate gradient method.

Compute

$$x_T := \underset{x \in K_T}{\text{argmin}} \ g(x), \tag{1}$$

where $K_T := \text{span}(b, Ab, \ldots, A^{T-1}b)$ is the Krylov’s subspace of order $T$ and

$$g(x) := \frac{1}{2} \left( \|e(x)\|_A^2 - \|x^*\|_A^2 \right) = \frac{1}{2} \|x\|_A^2 - b^T x.$$ 

return $x_T$.

Let $v_1, \ldots, v_T \in \mathbb{R}^n$ be an $A$-orthogonal basis for $K_T$. That is, we have that, for each $x \in K_T$, $x = \sum_{s=1}^{T} \alpha_s v_s$, for some $\alpha_1, \ldots, \alpha_T \in \mathbb{R}$; and $v_i \cdot_A v_j = 0$, if $i \neq j$, where $x \cdot_A y := x^T A y$

is the inner $A$-product.

(a) Show that the optimization problem (1) is equivalent to the following formulation

$$\underset{\alpha_1, \ldots, \alpha_T \in \mathbb{R}}{\text{argmin}} \ \sum_{s=1}^{T} \left( \frac{\alpha_s^2}{2} \|v_s\|_A^2 - \alpha_s b^T v_s \right). \tag{2}$$

(b) Argue that, given the $A$-orthogonal basis $v_1, \ldots, v_T$, we can solve problem (2) using only $T$ matrix-vector multiplications of $A$. 

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(c) Prove that one can compute the $A$-orthogonal basis $v_1, \ldots, v_T$ using only $O(T)$ matrix-vector and vector-vector multiplications.

Hint: Proceed in phases. In phase $s$, given an $A$-orthogonal basis $v_1, \ldots, v_{s-1}$ for $K_{s-1}$, extend it to an $A$-orthogonal basis $v_1, \ldots, v_{s-1}, v_s$ for $K_s$ by applying the Gram-Schmidt orthogonalization procedure to the vector $v'_s := Av_{s-1}$. (Why $K_s = \text{span}(v_1, \ldots, v_{s-1}, v'_s)$?) What can you say about $v_i \cdot A v'_s$, for each $i < s-2$?

Problem 3. (Understanding the lower end of the spectrum of a Laplacian matrix.) Let us fix an (unweighted) graph $G = (V, E, w)$ and let $L$ be its Laplacian matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

(a) Prove that all the eigenvalues of the Laplacian $L$ are non-negative, i.e., that $\lambda_1 \geq 0$.

(b) Show that $\lambda_1 = 0$ and the all-ones vector $\vec{1} := (1, \ldots, 1)$ is the corresponding eigenvector.

(c) Prove that, for any $k \geq 1$, $\lambda_k = 0$ iff $G$ has at least $k$ connected components.

Note: This means, in particular, that if $G$ is connected then $\lambda_2 > 0$.

Hint: The fact that we mentioned in class that, for any vector $x \in \mathbb{R}^n$, $x^T L x = \sum_{e = (u,v) \in E} (x_u - x_v)^2$ might be useful here.

Problem 4. (Bipartiteness and the value of $\lambda_n$.) Let $G = (V, E)$ be a bipartite graph and let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of its Laplacian. (A graph is bipartite iff one can partition its vertices into two sets $P$ and $Q$ such that each edge has one endpoint in $P$ and the other one in $Q$.)

(a) Show that whenever $G$ is $d$-regular (but not necessarily bipartite) we have that $\lambda_n \leq 2d$. (A graph is $d$-regular iff each vertex of $G$ has its degree equal to $d$.)

Note: One can show in a similar way that even when $G$ is not $d$-regular then $\lambda_n \leq 2d_{\text{max}}$, where $d_{\text{max}}$ is a maximum degree.

(b) Prove that for a $d$-regular graph $G$, if $G$ is bipartite then $\lambda_n = 2d$.

(c) (Extra credit) Let $G$ be $d$-regular and connected. Argue that if we have that $\lambda_n = 2d$ then $G$ is bipartite. Does this implication always hold if $G$ is not connected?