The Quorum Deployment Problem

Seth Gilbert

Grzegorz Malewicz

Outline

1. Background on quorum systems
2. The Quorum Deployment Problem
3. Hardness of quorum deployment
4. Special cases

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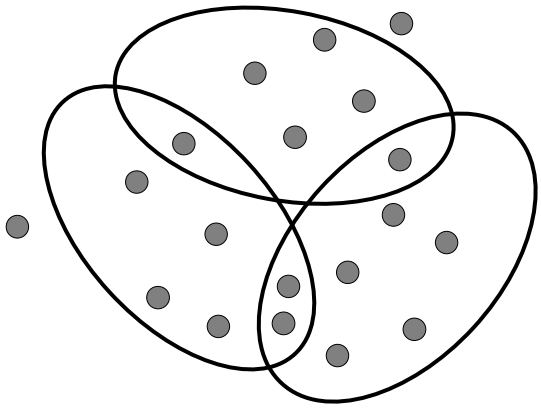
1. **Background on quorum systems**
2. The Quorum Deployment Problem
3. Hardness of quorum deployment
4. Special cases

Quorum Systems

- A *quorum system* is a collection of sets where all pairs of sets intersect.
- Notation
  - Q, a quorum system
  - $q \in Q$, a quorum
- Intersection
  - $q_1 \bigcap q_2 \neq \varnothing$

Example Quorum System



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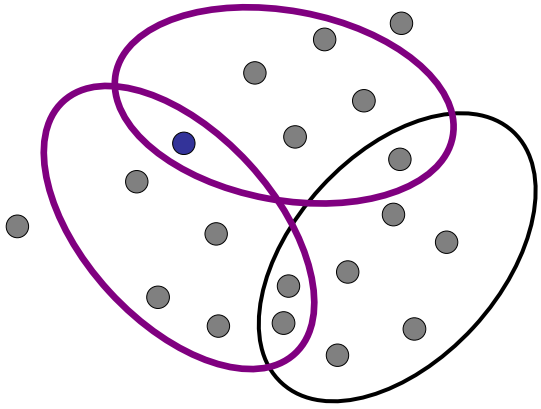

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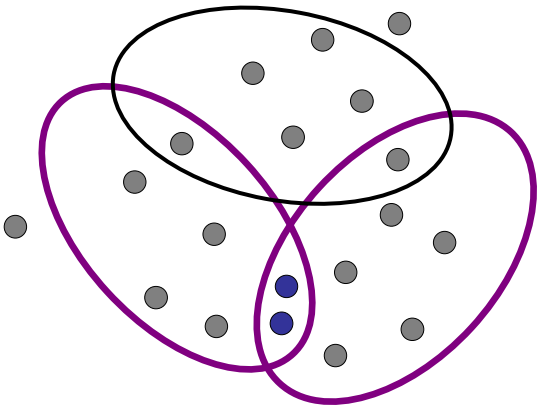
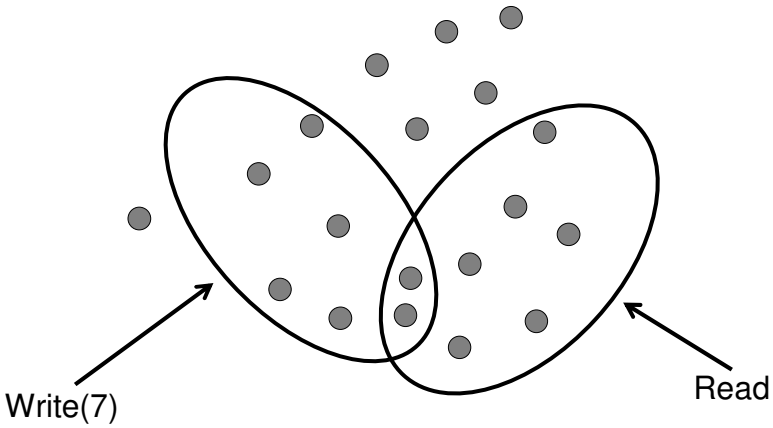


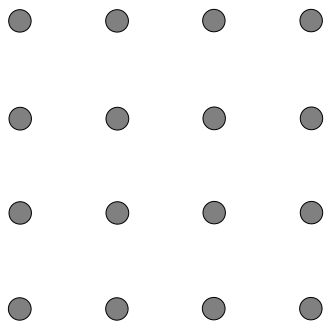
Using Quorums



Theoretical Quorum Systems

- Good properties
  - Fault Tolerance
  - Probe Complexity
  - Load
  - And others…
- Much research
  - Grid quorums
  - Hierarchical quorums
  - Paths quorums
  - etc.

Example: Grid



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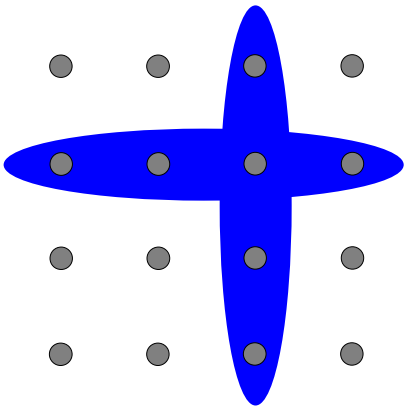

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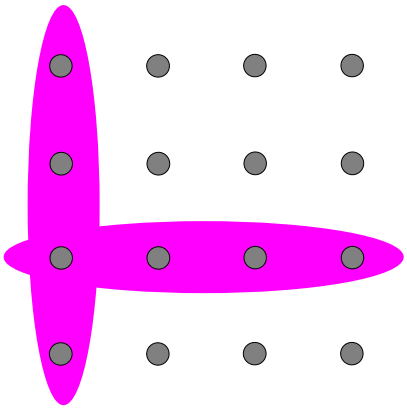

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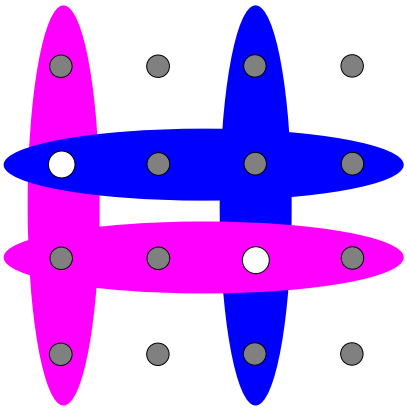

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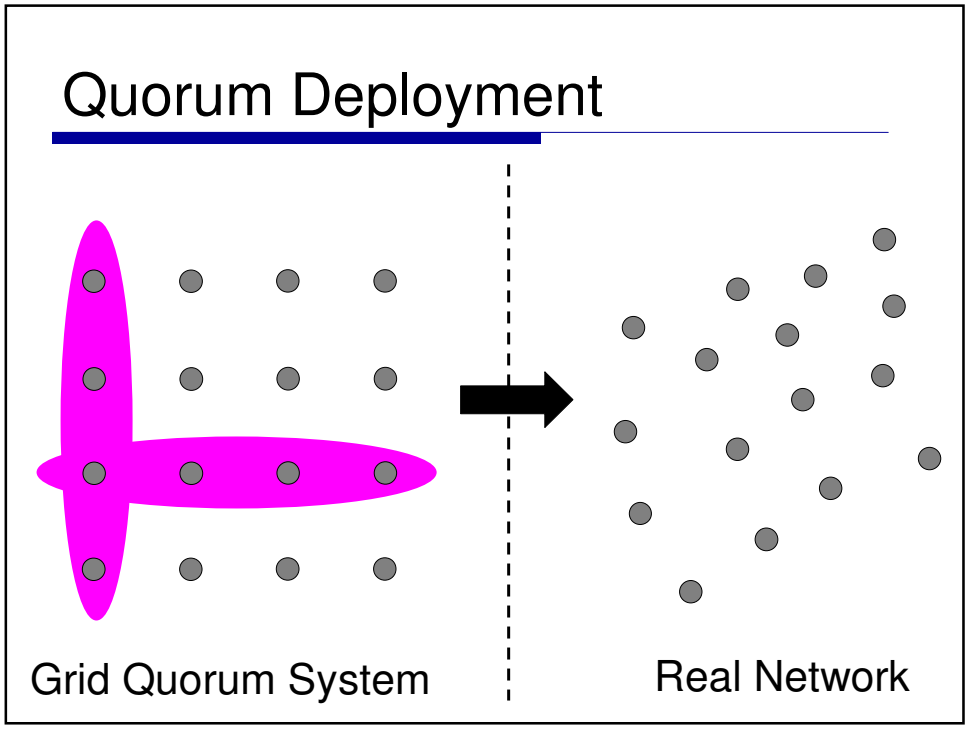
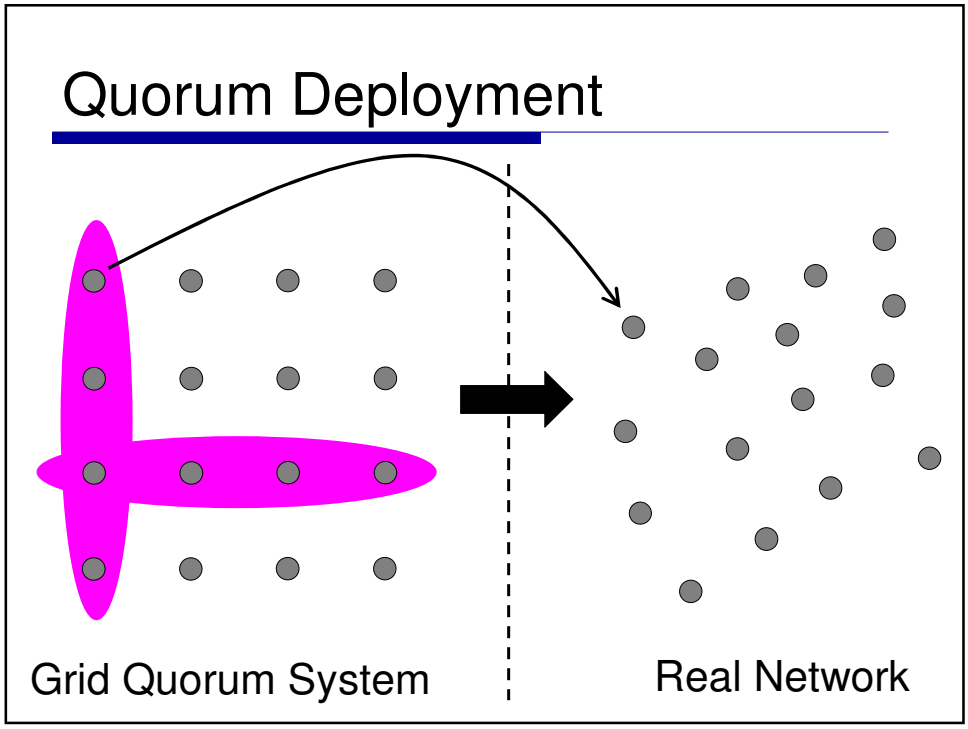
2. **The Quorum Deployment Problem**
3. Hardness of quorum deployment
4. Special cases


Quorum Deployment
Grid Quorum System
Real Network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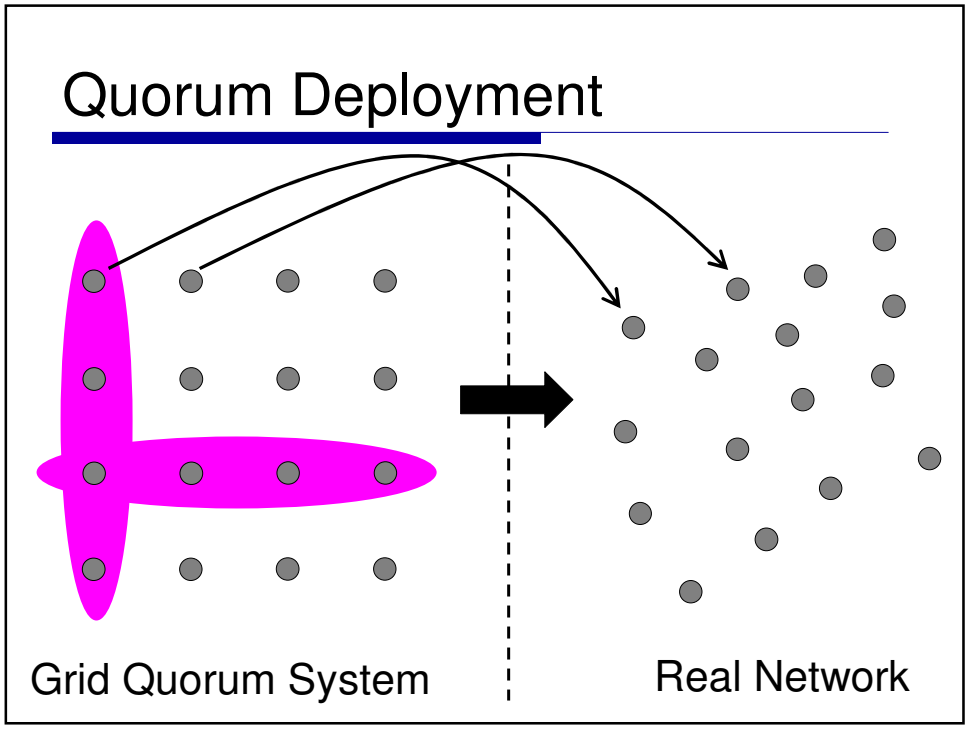
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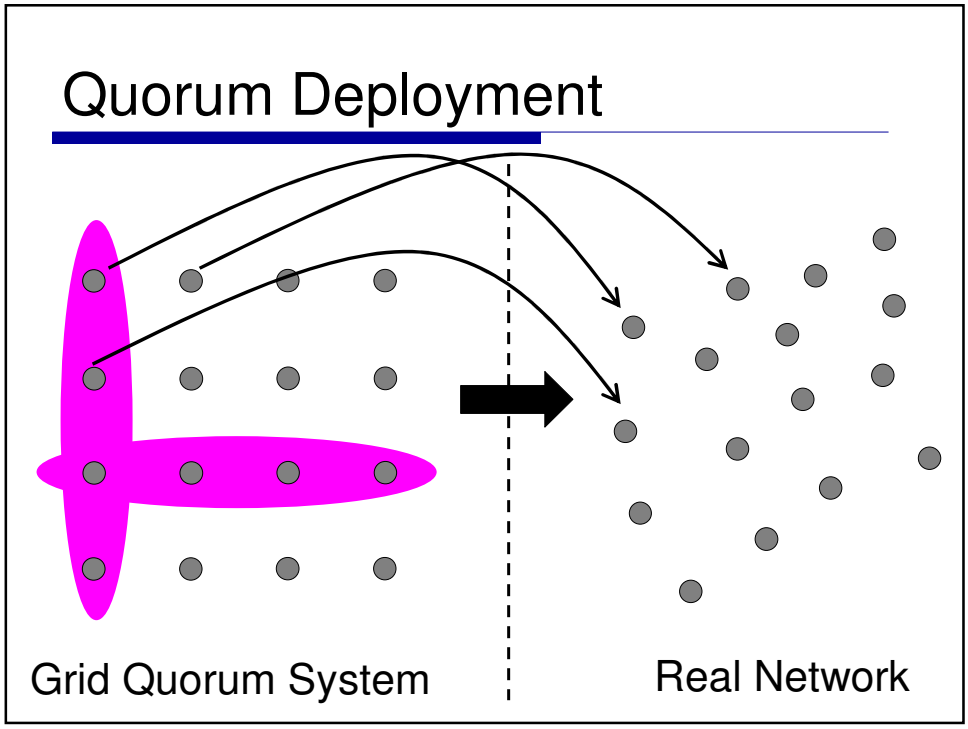
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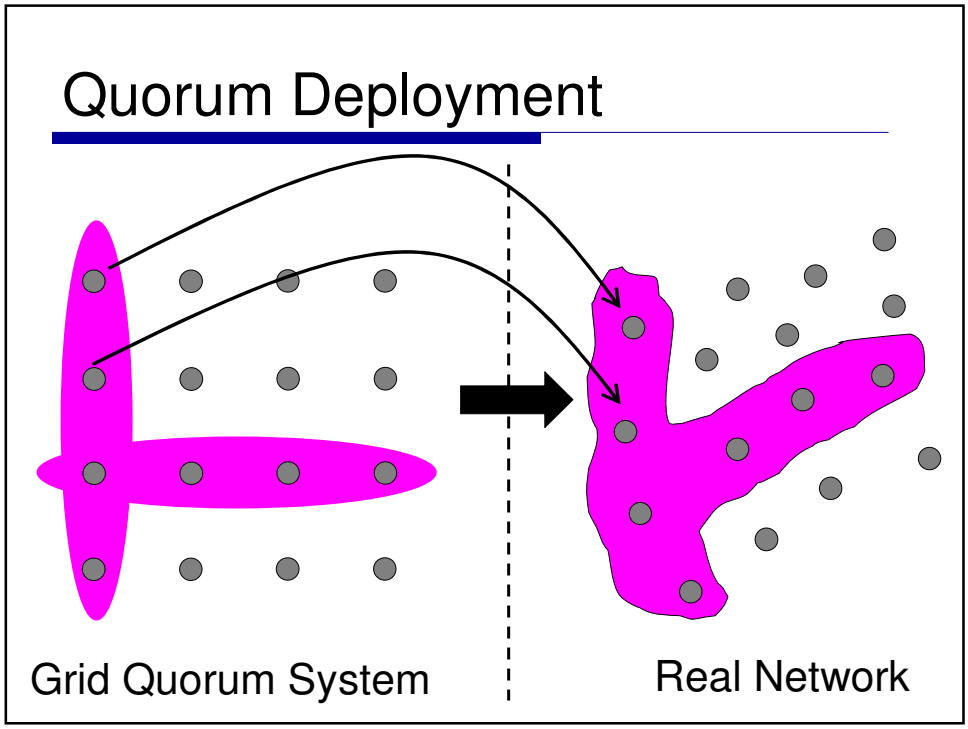
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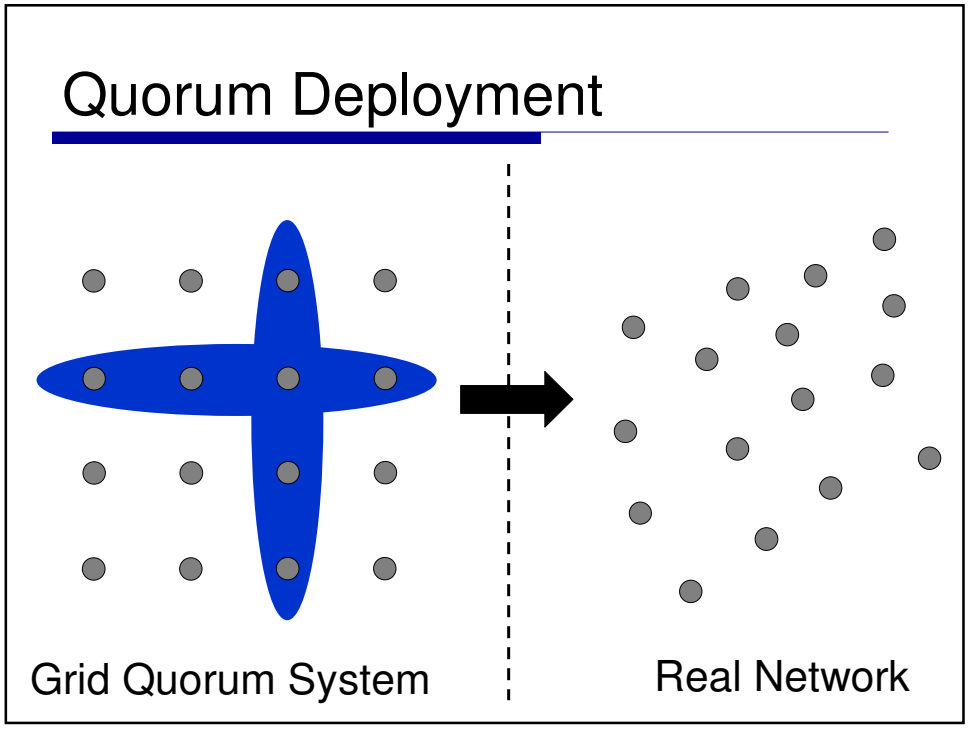
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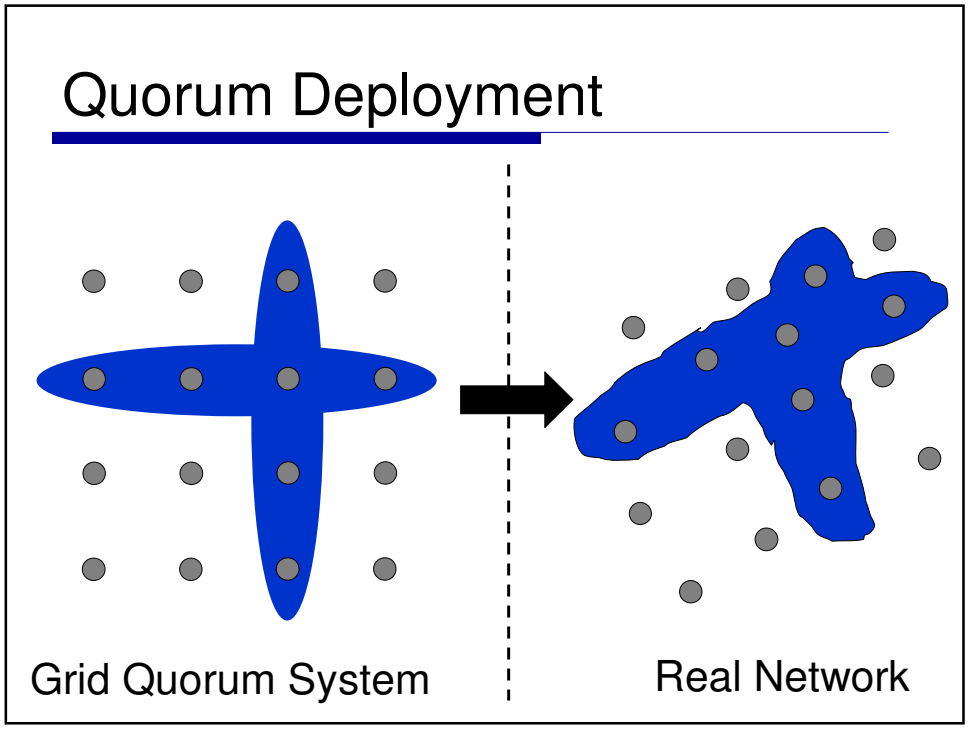
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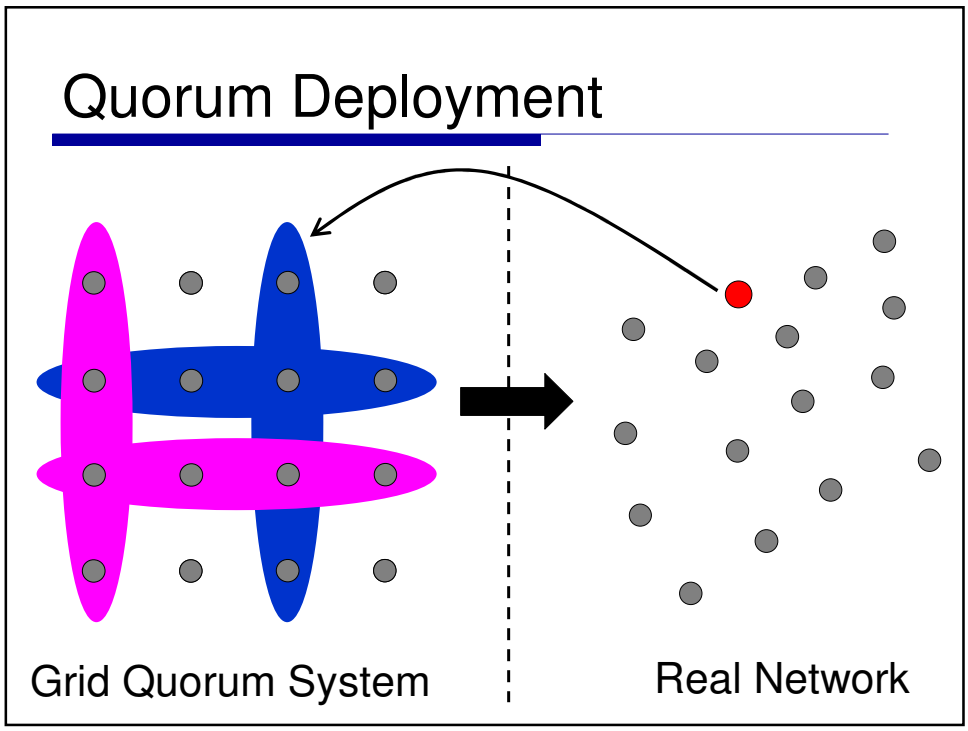
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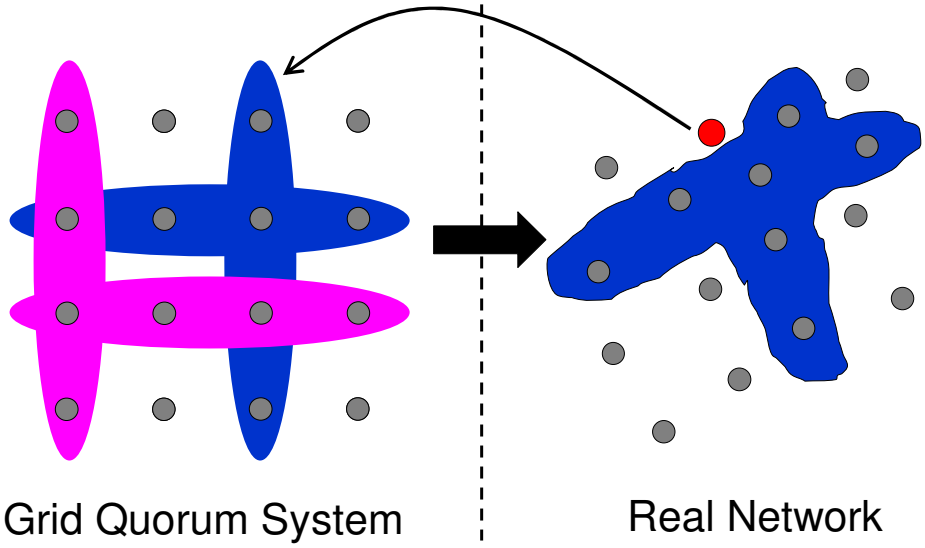
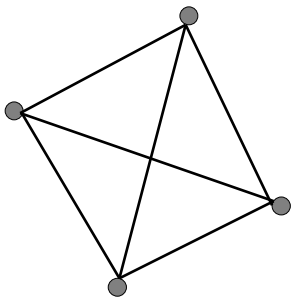
Quorum Deployment



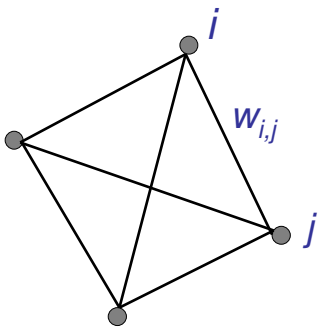
Quorum Deployment Problem

- Given:
  - Graph *G* with *n* vertices



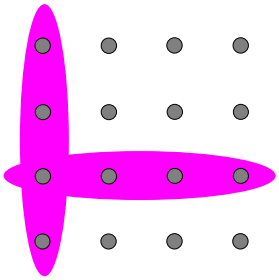
Quorum Deployment Problem

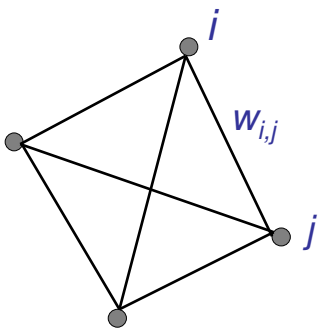
- Given:
  - Graph G with n vertices
  - Weighted edges; $w_{i,j}$ the weight of edge (i,j)



Quorum Deployment Problem

- Given:
  - Graph G with n vertices
  - Weighted edges; $w_{i,j}$ the weight of edge (i,j)
  - Quorum system Q





Quorum System Notation

- Quorum system, Q

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

$$Q(p, j) = \begin{cases} 0 \text{ if node } j \text{ is } \textbf{not} \text{ in quorum } p \\ 1 \text{ if node } j \text{ is in quorum } p \end{cases}$$

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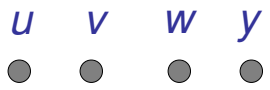

$$\begin{matrix} u & v & w & y \end{matrix}$$
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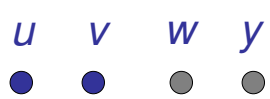

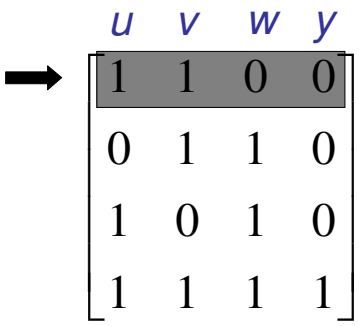

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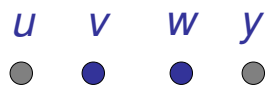

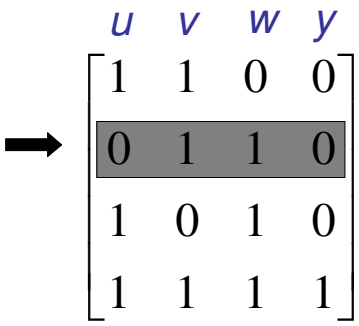

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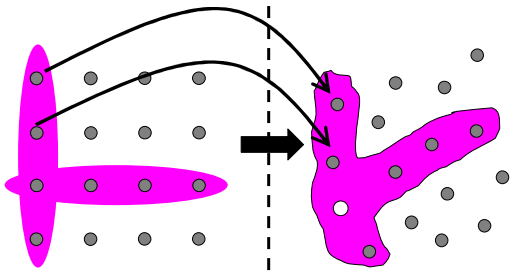
Quorum Deployment Problem

- Given:
  - Graph G with n vertices
  - Weighted edges; $w_{i,j}$ the weight of edge (i,j)
  - Quorum system Q
- Output:
  - Optimal deployment of Q for graph G

Optimal Quorum Deployment

- Two parts
  - β, mapping quorum system to real nodes
  - α, mapping from nodes to quorums



Optimal Quorum Deployment

- Two parts
  - β, mapping quorum system to real nodes
  - α, mapping from nodes to quorums
- Minimize the deployment cost:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,j} \cdot Q(\alpha(i), \beta(j))$$

Optimal Quorum Deployment

- Two parts
  - β, mapping quorum system to real nodes
  - α, mapping from nodes to quorums
  - Similar to Quadratic Assignment Problem
    - 2 degrees of freedom, instead of 1
    - QAP only has one permutation

Optimal Quorum Deployment

- Two parts
  - β, mapping quorum system to real nodes
  - α, mapping from nodes to quorums
- Preserve good properties of quorum system
  - fault tolerance, availability, probe complexity

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3. **Hardness of quorum deployment**
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Main Result

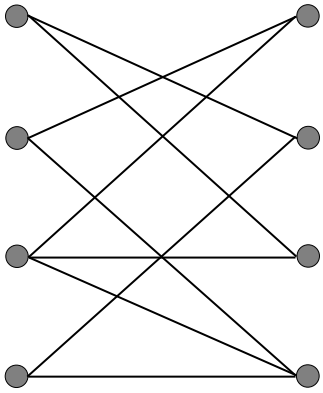
- **Theorem 1**:

  For any $\delta > 0$, it is NP-hard to approximate the optimal Quorum Deployment within a factor of n^{δ}.

Balanced Complete Bipartite Subgraph

- Given:
  - Bipartite graph, $G = (V,E)$
    - Left nodes, L
    - Right nodes, R
  - parameter, k
- Output:
  - Complete, bipartite subgraph, G'
    - k left nodes
    - k right nodes

Balanced Complete Bipartite Subgraph



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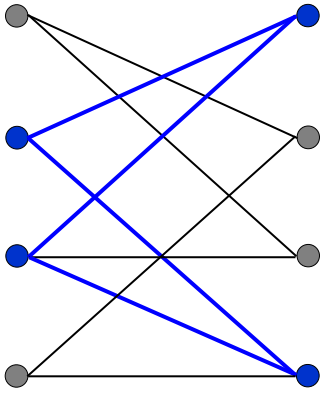

Reduction: BCBS → Deployment

- Given BCBS, {*G=(V,E), k*}
- Output instance of Quorum Deployment
  - *Q, w,* $n = |V| + 1$
  - Gap creating:

$$(G,k) \in \text{BCBS} \Rightarrow \exists \alpha, \beta : \text{deploy cost} \le n^2$$

$$(G,k) \notin \text{BCBS} \Rightarrow \forall \alpha, \beta : \text{deploy cost} > n^x$$

Reduction: BCBS → Deployment

$$w_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \text{ and } i, j < n \\ n^x & \text{if } (i,j) \notin E \text{ and } i, j < n \\ 1 & \text{if } (i = n) \text{ or } (j = n) \end{cases}$$

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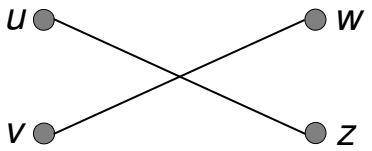

	u	v	w	z	
u	n^x	n^x	n^x	1	1
v	n^x	n^x	1	n^x	1
w	n^x	1	n^x	n^x	1
z	1	n^x	n^x	n^x	1
	1	1	1	1	1

Reduction: BCBS → Deployment

$$Q(p,i) = \begin{cases} 1 & \text{if } p,i \le k \\ 1 & \text{if } i = n \\ 1 & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$$

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$$Q(p,i) = \begin{cases} 1 & \text{if } p,i \le k \\ 1 & \text{if } i = n \\ 1 & \text{if } p = n \\ 0 & \text{otherwise} \end{cases}$$

$$k = 2 \quad \Rightarrow \quad Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Proof: Case 1

- Assume there exists a bipartite complete subgraph of size *k*
- Construct a deployment:
  - map complete subgraph to the "all 1's" corner
  - map last row/column to itself
- Resulting cost: $k^2+2n-1 \le n^2$
  - k^2 cost for "all 1's" corner
  - $2n-1$ cost for the last row/column

Proof: Case 2

- Assume **no** complete bipartite subgraph of size *k*
  - Every deployment must include an expensive edge. Otherwise, we can find a complete bipartite subgraph

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Metric Cost Deployment

- **Theorem 2**:

  If the weight is a *distance metric* (*symmetric* and *triangle inequality*) then it is NP-hard to find an optimal deployment.

Metric Cost Deployment

- **Theorem 2**:
  If the weight is a *distance metric* (*symmetric* and *triangle inequality*) then it is NP-hard to find an optimal deployment.
- Proof:
  Same reduction, except metric weights.

$$\left[\begin{array}{cc|ccc} 2 & 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 \\ \hline 2 & 1 & 2 & 2 & 1 \\ 1 & 2 & 2 & 2 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

Relaxed Metric Cost Deployment

- Relax assumptions that quorums intersect
  - Consider a quorum system to be a collection of sets.
- Assume the weight is a metric (symmetric and triangle inequality).

Relaxed Metric Cost Deployment

- **Theorem 3**:

  For any $\delta > 0$, it is NP-hard to approximate the optimal Metric Cost Deployment within a factor of n^{δ}.

- Proof ideas:
  - Reduction from 3-Partition Problem
    - Partition graph into pieces
  - Extension from Queyranne'86 reduction for the Quadratic Assignment Problem

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4. **Special cases**
   - Bounded Cost Networks
   - Partial Deployment
   - Hyperbolic Quorum Systems

Bounded Cost Networks

- Assume that U is an upper bound on the distance between two nodes.
- Assume that L is a lower bound on the distance between two nodes.
- Trivial: all deployments are a $\frac{U}{L}$ approximation of optimal.

Partial Quorum Deployment

- Recall: deployment consists of two parts
  - β, mapping quorum system to real nodes
  - α, mapping from nodes to quorums
- Assume quorum system specification includes α.
  - E.g., node 1 uses quorum 1, node 2 uses quorum 2, etc.

Partial Quorum Deployment

- **Theorem 4:**

  We can find an optimal Partial Deployment in polynomial time.

Partial Quorum Deployment

- **Theorem 4:**

  We can find an optimal Partial Deployment in polynomial time.

- Proof:
  - Construct bipartite graph — matching in graph is a permutation.
  - Weight of edge is cost of assigning node to that quorum .
  - Find minimum weight matching.

Deployable Quorum Systems

- Are there quorum systems that are easy to deploy (for all networks)?

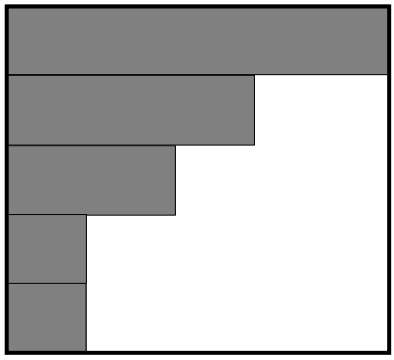
Deployable Quorum Systems

- Are there quorum systems that are easy to deploy (for all networks)?
  - Majorities:
    - A quorum is a majority of nodes in the system
    - Impossibility results do not apply since it is not a quorum system, by our formal definition.
  - Others??

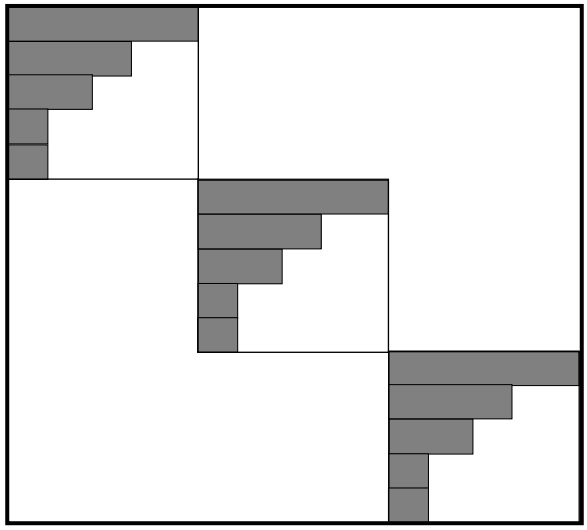
Hyperbolic Quorum Systems

- Advantages:
  - (Approximately) efficient to deploy
  - Help to understand complexity of quorum systems
- Disadvantages
  - NP-hard to deploy optimally
  - Not fault tolerant

Hyperbolic Quorum Systems



Hyperbolic Quorum Systems



Hyperbolic Quorum Systems

- **Theorem 5:**

  We can find an approximately optimal deployment for a hyperbolic quorum system.

Hyperbolic Quorum Systems

- **Theorem 5:**

  We can find an approximately optimal metric cost deployment for a hyperbolic quorum system.

- Note:

  The running time and the approximation depend on the complexity of the quorum system.

Open Questions

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- Can we deploy:
  - Grid quorum systems
  - Paths quorum systems
  - Others?

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- Can we deploy:
  - Grid quorum systems
  - Paths quorum systems
  - Others?
- Can we deploy quorum systems in:
  - growth-restricted metrics
  - metric spaces (if they intersect)
  - on a linear network

Open Questions

- Can we develop a quorum system with good fault tolerance, good probe complexity, and good load — that can also be deployed efficiently?

Conclusions

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If you want to use quorums, use majorities — or other simple, deployable quorum systems.

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If you want to ***use*** quorums, use majorities — or other simple, deployable quorum systems.

If you want to ***develop*** quorums, think about quorum deployment.