Addition

Basic component: Full adder - combinational

\[ x \rightarrow FA \rightarrow c = \text{MSB}(x+y+z) = \text{majority} \]
\[ y \rightarrow FA \rightarrow s = \text{LSB}(x+y+z) = \text{parity} \]

Problem:
Add 2 N-bit numbers

Ripple-carry adder

\[ a_4 b_4 a_3 b_3 a_2 b_2 a_1 b_1 \]

N-bit #\'s \( \Rightarrow \Theta(N) \) time, \( \Theta(N) \) HW, combinational

Serial adder

\( \Theta(N) \) time, \( \Theta(1) \) hardware, sequential (clocked)
Fast addition

Idea: carries are the hard part.
Know carries ⇒ compute sum in Θ(1) time

\[
\begin{array}{ccccccc}
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\hline
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
g & p & g & k & p & p & g (k)
\end{array}
\]

Classify stages:
- **kill**: 0 ⇒ carry-out = 0
- **propagate**: 0 or 1 ⇒ carry-out = carry-in
- **generate**: 1 ⇒ carry-out = 1

Carry into stage = \{ 1 if most recent non-\(p\) is \(k\), 0 otherwise \}

When do 2 consecutive stages kill, prop, gen?

\[
\begin{array}{c|ccc}
x_i \otimes x_{i+1} & k & p & g \\
\hline
k & k & k & g \\
x_i & p & k & p & g \\
g & k & g & g & g
\end{array}
\]

Associative!
Theorem. Let \( x_i \) be carry status of stage \( i \), where \( x_0 = k \). Define \( y_i = x_0 \otimes x_1 \otimes \ldots \otimes x_i \).

Then
\[
\begin{align*}
y_i = k & \implies c_i = 0 \\
y_i = g & \implies c_i = 1 \\
y_i = p & \text{ does not occur}
\end{align*}
\]

Proof. Induction on \( i \). \( \square \)

Log-time circuit:
\[
\begin{align*}
y_0 &= x_0 \\
y_1 &= x_0 \otimes x_1 \\
y_2 &= x_0 \otimes x_1 \otimes x_2 \\
y_N &= x_0 \otimes x_1 \otimes \ldots \otimes x_N
\end{align*}
\]

Use tree for each calculation:

Use tree to broadcast inputs (bounded-degree network):

\[
\begin{align*}
\text{Time} &= \Theta(\log N), \\
\text{HW} &= \Theta(N^2).
\end{align*}
\]
**Carry-lookahead addition**

$\Theta(\log N)$ time, $\Theta(N)$ HW.

"Parallel prefix"

Let $[i, j]$ denote $x_i \otimes x_{i+1} \otimes \ldots \otimes x_j$

**Lemma.** $[i, j] \otimes [j+1, k] = [i, k] \otimes$

$x_i = [i, i]$

$y_i = [0, i]$

**Build tree:**

Globally:

Left child values are passed up.
Similar method:

Left child values are passed up and right

Postscript Kill, propagate, generate first used in standard relay calculator circa mid-1940's.

O(1)-time addition (in their model)