

## 1 Overview

In this lecture we discuss the Hamiltonian cycle and path problems, with an emphasis on grid graphs, and use these problems to prove some NP-hardness results for games and lawn mowing.

## 2 Hamiltonian Cycle and Path

A *Hamiltonian cycle* (also *tour*, *circuit*) is a cycle visiting each vertex exactly once. Graphs are said to be *Hamiltonian* if they contain a Hamiltonian cycle. A *Hamiltonian path* is a path visiting each vertex exactly once. The decision problems ask whether a Hamiltonian cycle or path exists in a given graph.

Hamiltonicity is named after William Rowan Hamilton, an Irish mathematician, who studied Hamiltonian cycles on the dodecahedron. Hamilton commercialized his study as the Icosian Game (so named because the dodecahedron is the dual of the icosahedron).



## 2.1 NP-complete restrictions

The following cases were stated briefly in lecture, without working through the proofs:

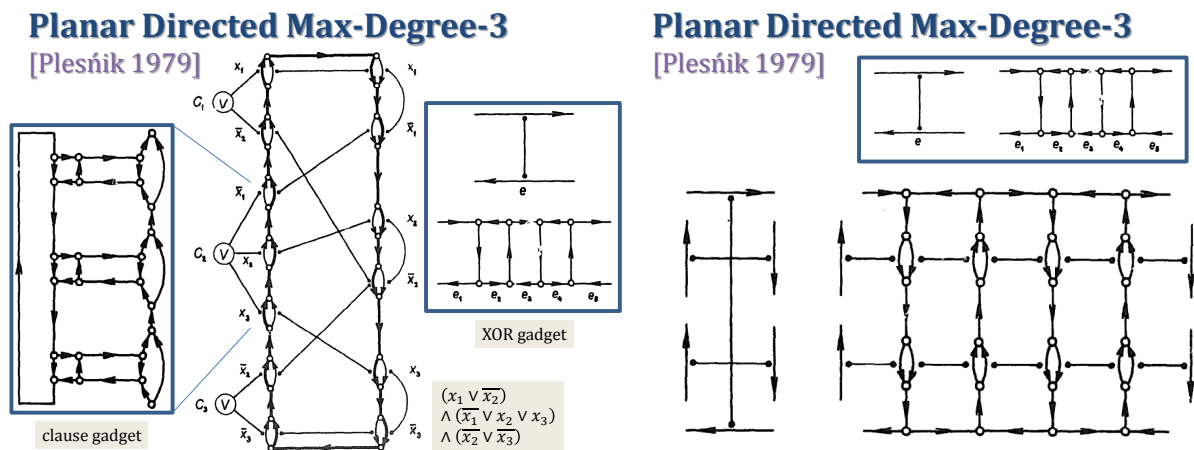
- Hamiltonian path with specified start and end vertices, by reduction from Hamiltonian cycle (add a vertex incident only on the start and end vertices, which must then be in the cycle and so removing it results in a Hamiltonian path) [GJ79]
- planar 3-regular 3-connected graphs with minimum face degree 5 [GJT76]
- bipartite graphs [Kri75]
- Hamiltonian cycle on squares of graphs, where the square of a graph is the graph with edges added between all vertices connected by a path of length 2 [?]
- Hamiltonian cycle given a Hamiltonian path (i.e., knowing a path does not in general help when finding a cycle) [PS76]

## 2.2 Polynomial-time restrictions

These graphs are always Hamiltonian, so the problem is easy:

- planar 4-connected graphs [Tut56]
- cubes of graphs, where the cube of a graph is the graph with edges added between all vertices connected by a path of length 3 or less [Kar68]

## 2.3 Planar directed max-degree-3



Hamiltonian cycle on planar directed max-degree-3 graphs is NP-complete by reduction from 3SAT [Ple79]. The reduction relies on an XOR gadget used to enforce that exactly one of two edges is in the cycle. Variables are then represented as pairs of doubled edges linked by a XOR gadget, forcing the pairs to have opposite membership, representing true and false assignments. The clause gadget

contains a large cycle which can be in the Hamiltonian cycle only if one of the incoming literals (connected to the clause by XOR gadgets) is true. This reduction also requires a crossover gadget to allow the XOR gadgets connecting variables to clauses to cross; the crossover gadget itself uses the XOR gadgets, essentially exploding them.

## 2.4 Planar undirected bipartite max-degree-3

### Planar Bipartite Max-Degree-3

[Itai, Papadimitriou, Szwarcfiter 1982]



Hamiltonian cycle on planar undirected bipartite max-degree-3 graphs is NP-complete by reduction from the corresponding directed graph problem [IPS82]. In a max-degree-3 directed graph, each vertex has either in-degree or out-degree 1, so that edge must be in any Hamiltonian cycle. Then we can map each vertex in the directed graph to a pair of vertices in the undirected graph. Any Hamiltonian cycle in the resulting graph will alternate between vertices from the original graph and vertices introduced to model forced edges, so the resulting graph is bipartite.

## 3 Hamiltonicity in Grid Graphs

### Grid Graphs

[Itai, Papadimitriou, Szwarcfiter 1982]

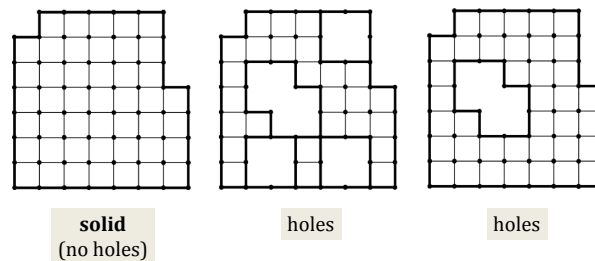


figure by Arkin, Fekete, Islam, Meijer, Mitchell, Núñez-Rodríguez, Polishchuk, Rappaport, Xiao 2009

*Grid graphs* are graphs with vertices on a (subset of a) lattice and edges between all pairs of vertices

with unit distance. Unqualified, grid graphs are usually on the square lattice, but the triangular and hexagonal lattices can also be considered.

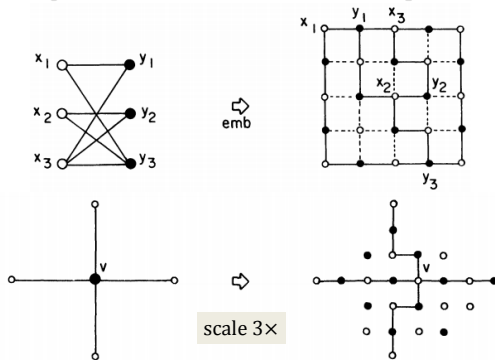
Faces of grid graphs of unit area are called *pixels*, while faces with greater area (i.e., containing at least one lattice point that is not a vertex) are called *holes*. *Solid* grid graphs have no holes.

### 3.1 Hamiltonian cycle on grid graphs

Hamiltonian cycle on solid grid graphs is polynomial time [UL97].

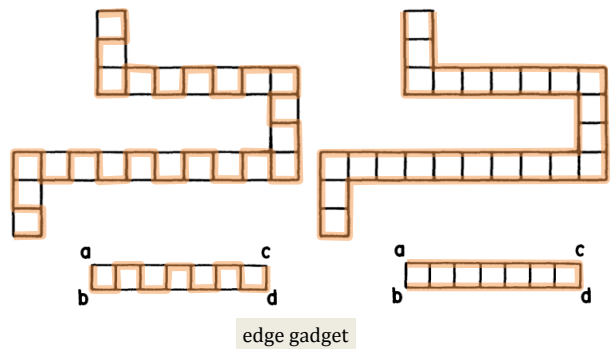
#### Planar Bipartite Graph Drawing

[Itai, Papadimitriou, Szwarcfiter 1982]



#### Hamiltonicity in Grid Graphs

[Itai, Papadimitriou, Szwarcfiter 1982]



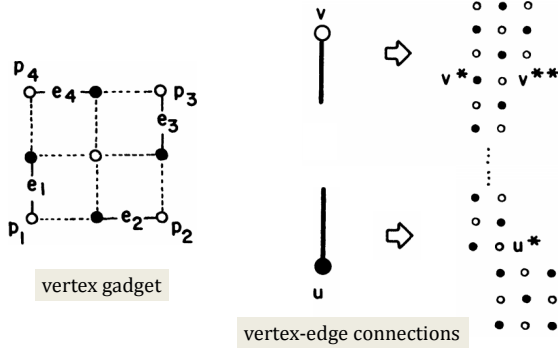
Hamiltonian cycle on grid graphs containing holes is NP-complete by reduction from Hamiltonian cycle on planar undirected bipartite max-degree-3 graphs [IPS82]. The reduction first embeds the input bipartite graph on a grid graph using the grid graph's inherent 2-coloring. Any parity violations can be sidestepped by scaling the grid graph by a factor of 3, allowing "wiggles" to be added as required.

The Hamiltonian cycle on the input graph need not use every edge, but the Hamiltonian cycle on the output grid graph needs to visit each vertex, so our edge gadget can be traversed in two ways, a "zigzag" corresponding to using the edge (a Hamiltonian path from one end of the gadget to the other) and a border traversal corresponding to not using the edge (a Hamiltonian path that returns to the same end of the gadget).

The vertex gadget is a 3-by-3 square of vertices, the corners of the gadget having the parity of the corresponding vertex in the input graph. This gadget has Hamiltonian paths from every corner to every other corner that traverse four marked edges. Edge gadgets connect to white vertices flush with one of the corners, but connect to black vertices offset by one (*pin joint*); this arrangement allows the edge to be used or not while preserving the parity of the cycle.

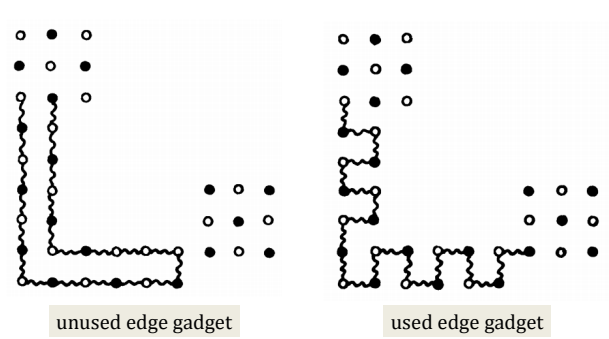
## Hamiltonicity in Grid Graphs

[Itai, Papadimitriou, Szwarcfiter 1982]



## Hamiltonicity in Grid Graphs

[Itai, Papadimitriou, Szwarcfiter 1982]



### 3.2 Euclidean traveling salesman

Given a set of points in the plane, the Euclidean traveling salesman problem asks for a tour through the points with Euclidean length less than  $k$  (or in the optimization problem, with minimum length). This problem is NP-hard by reduction from Hamiltonian cycle on grid graphs. The reduction simply passes the grid graph's vertices as the points in the ETSP problem, where the desired length  $k$  is the number of points; if there is such a tour, it must be a cycle in the grid graph because choosing any nonadjacent pair of points will prevent meeting the length bound, and if a Hamiltonian cycle exists in the grid graph, it is a valid tour.

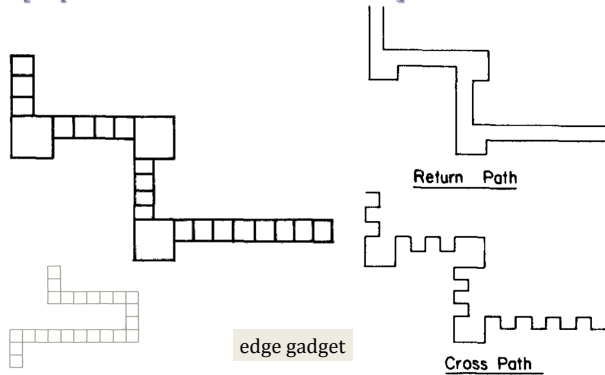
### 3.3 Platform games with collectibles and timers

Platform games whose objective is to collect all collectibles (e.g., coins) in a level before a timer expires are NP-hard by reduction from Hamiltonian cycle on grid graphs [For10]. The reduction simply places coins at all the grid graph vertices and sets the timer such that moving between any nonadjacent pair of points (i.e., not taking the minimum tour) will result in the timer expiring.

### 3.4 Max-degree-3 grid graphs

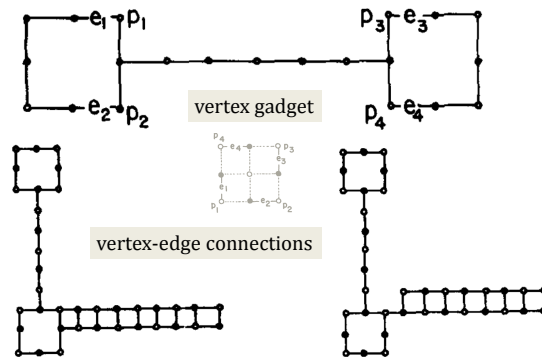
#### Max-Degree-3 Grid Graphs

[Papadimitriou & Vazirani 1984]



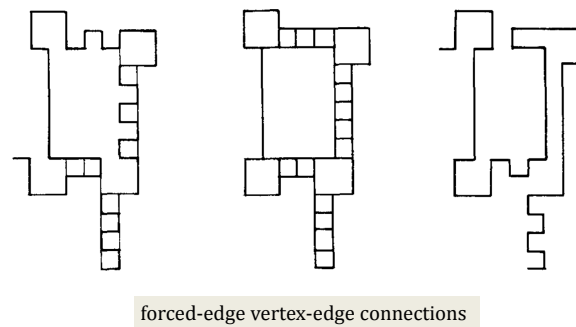
#### Max-Degree-3 Grid Graphs

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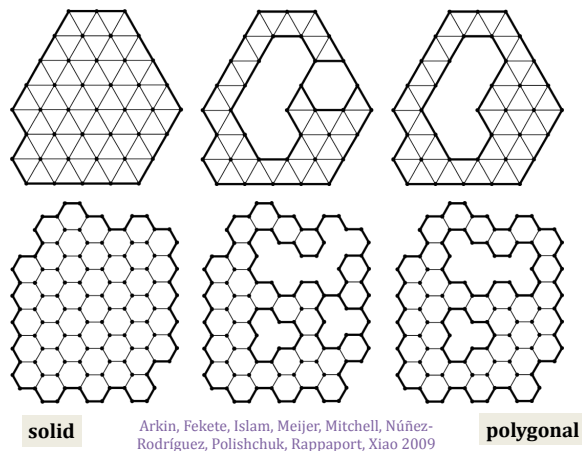


Hamiltonian cycle in max-degree-3 grid graphs is NP-complete by a similar reduction for unconstrained grid graphs, with the gadgets modified to avoid degree-4 vertices [PV84]. The edge gadget gains holes at corners, but has the same topology with two configurations. Vertex gadgets are now "dumbbells". Degree-2 vertices have their edge gadgets connected to opposite ends of the dumbbell; degree-3 vertices have the forced edge connected to both ends of the dumbbell via a fork gadget. (This fork gadget is required to preserve parity.)

### 3.5 Euclidean degree-3 minimum spanning tree

Minimum spanning tree is usually polynomial, but if bound to degree-2 it's the traveling salesman problem. Euclidean degree-3 minimum spanning tree is still hard by reduction from Hamiltonian path on max-degree-3 grid graphs [PV84]. The reduction adds new vertices very close to each existing vertex in the grid graph, forcing the edge between that vertex and the new vertex to be in the minimum spanning tree. The remainder of the tree is then finding a Hamiltonian path in the grid graph.

### 3.6 Grid graph Hamiltonicity taxonomy



*Superthin* grid graphs have no pixels; all faces are holes or the outside face. *Thin* grid graphs have all vertices on the boundary. *Polygonal* grid graphs have no shared edges between holes or the outside face; polygonal graphs could be considered anti-superthin. With these definitions we can taxonomize Hamiltonicity over various kinds of graphs.

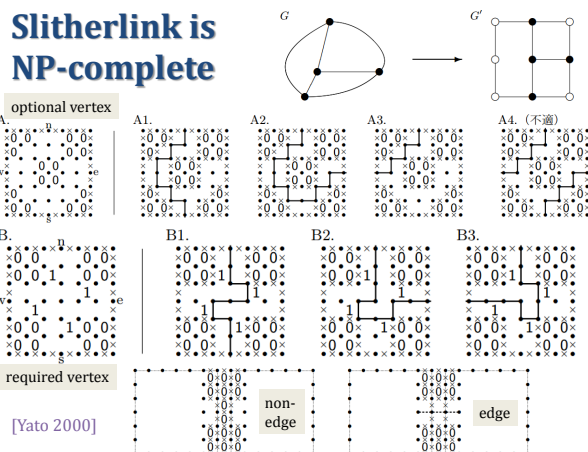
	triangular	square	hexagonal
solid	poly	poly	open
superthin	NP-hard	NP-hard	open
max-degree-4	NP-hard	-	-
max-degree-3	NP-hard	NP-hard	NP-hard
polygonal	poly	open	NP-hard

## 4 Reductions from Hamiltonicity

### 4.1 Settlers of Catan Longest Road card

Mate-in-1 and mate-in-0 are NP-complete for Settlers of Catan using the Longest Road card by reduction from Hamiltonian cycle in hexagonal grid graphs [?]. The Longest Road card gives the player with the longest road two victory points. If the opponent has a road of length  $n - 1$  then mate-in-1/0 requires finding/having a Hamiltonian path in the hexagonal grid (with the opponents' roads forming obstacles).

## 4.2 Slitherlink



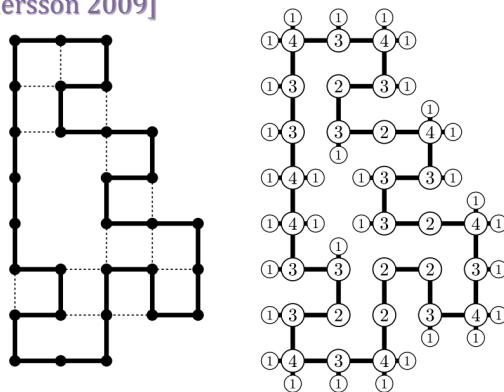
Slitherlink is a Nikoli game played on a grid graph in which some pixels are labeled with the numbers 0 through 4. The objective of the game is to find a cycle (not necessarily Hamiltonian) along the grid lines such that the numbered pixels are bordered by that number of edges in the cycle.

Slitherlink is NP-complete by reduction from Hamiltonian cycle on grid graphs [Tak00]. There are two vertex gadgets, one for vertices that are in the input graph (which must be in the cycle) and one for vertices in the grid graph that were not in the original graph (which are optional).

## 4.3 Hashiwokakero

### Hashiwokakero is NP-Complete

[Andersson 2009]



Hashiwokakero is a Nikoli game in which the goal is to add orthogonal noncrossing (multi-)edges to connect a given set of vertices with specified degree. Hashiwokakero is NP-complete by reduction from Hamiltonian cycle in grid graphs [And09]. Each vertex of the grid graph is mapped to a vertex with specified degree  $2 + b$  where  $b$  is the number of directions in which it is not connected to another vertex. Those directions are filled with vertices of specified degree 1, which must fill



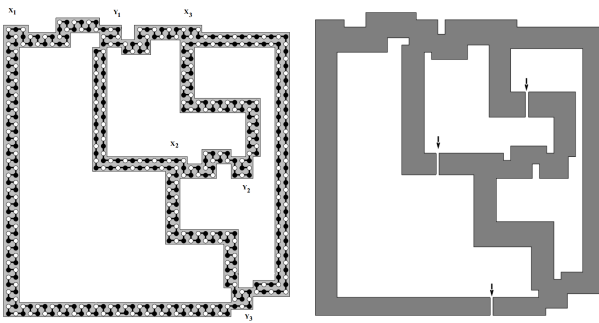
up the  $b$  part of their neighboring vertex. The leftover 2 edges per vertex that must be placed are exactly the Hamiltonian cycle in the grid graph.

#### 4.4 Optimal lawn mowing and milling

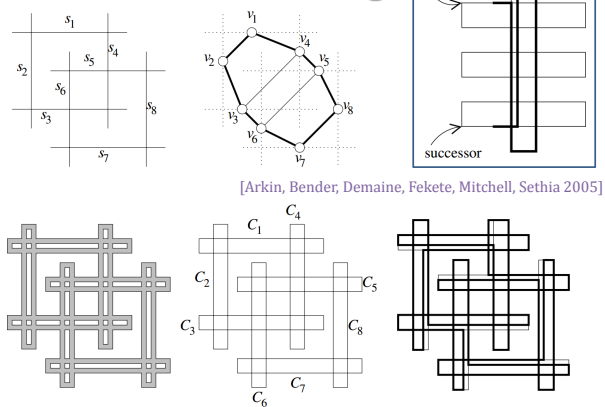
Lawn mowing and milling problems both involve cutting a specified region with a tool. In lawn mowing problems the tool path can go outside the region, while in milling it must stay inside. The goal is generally to find the shortest path. Milling problems arise in actual physical milling, while lawn mowing problems arise in laser and waterjet cutting and in each layer of 3D printing by deposition.

Milling and lawn mowing are NP-hard for grid polygons and a unit square tool by reduction from Hamiltonicity in grid graphs [AFM00]. Minimum-turn milling (also infinite-acceleration milling) is NP-complete by reduction from Hamiltonicity in unit orthogonal segment intersection graphs [ABD<sup>+</sup>05]; this problem arises in physical milling where the cost of milling a straight line is insignificant compared to the cost of stopping to turn.

##### Milling & Lawn Mowing [Arkin, Fekete, Mitchell 2000]



##### Minimum-Turn Milling



## References

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