6.892: Algorithmic Lower Bounds, Spring 2019

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Problem Set 3 Solution

Due: Tuesday, February 26, 2019 at noon

Problem 3.1 [Set Splitting]. Prove that the following problem is NP-complete.

> Set Splitting: Given a finite set S and a collection C of subsets of S, is there a partition of S into disjoint sets S_1 and S_2 such that no set in C is a subset of S_1 or S_2?

Hint: The reduction is straightforward if you choose the right problem to reduce from.

Solution: We begin by showing that Set Splitting is in NP. Let (S, C) be a Set Splitting instance. We nondeterministically construct the sets S_1 and S_2 by guessing which set to put each element of S in. Then we verify that every set in C is neither a subset of S_1 or of S_2. This algorithm takes linear time for guessing and linear time for verifying, so it is polynomial-time. Thus Set Splitting is in NP.

We now prove that Set Splitting is NP-hard by reducing from the NP-hard problem Positive NAE 3-SAT. Let (X, φ) be a Positive NAE 3-SAT instance, where X is a set of variables and φ is a formula which is a conjunction of positive NAE clauses over those variables. Define C as follows: for every clause (x_1, x_2, x_3) in φ, add the set $\{x_1, x_2, x_3\}$ to C. Our reduction outputs the Set Splitting instance (X, C) in linear time.

We prove that the Positive NAE 3-SAT and Set Splitting instances are equivalent. Define a bijection between truth-value assignments of X and bipartitions of X as follows: an assignment of truth values to the variables of X corresponds to a partition of X into disjoint sets S_1, S_2 where S_1 is the set of true-valued variables and S_2 is the set of false-valued variables. Then an assignment of truth values satisfies φ if and only if the corresponding partition is a splitting of C. This is because each clause (x_1, x_2, x_3) of φ is satisfied under a given truth assignment if and only if the corresponding set $\{x_1, x_2, x_3\}$ of C is split between the S_1 and S_2 corresponding to that truth assignment. Therefore our reduction is correct.

Because Positive NAE 3-SAT is NP-hard, so is Set Splitting. Thus Set Splitting is both NP-hard and in NP, so it is NP-complete.