| 6.890 | Lecture 11 | Oct. 9, 2014 |
|---|---|---|

## Recall from L10:

**L-reduction:**

$$\begin{array}{ccc} A & \longrightarrow & B \\ x & \xrightarrow{f} & x' = f(x) \\ & & \vdots \\ g(x, y') = y & \xleftarrow{g} & y' \end{array}$$

① $OPT_B(x') = O(OPT_A(x))$ $\leq \alpha \cdot$

② $|cost_A(y) - OPT_A(x)| = O(|cost_B(y') - OPT_B(x')|)$ $\leq \beta \cdot$

[Papadimitriou & Yannakakis - JCSS 1991]

$\Rightarrow$ PTAS-reduction

- for minimization: $\delta(\varepsilon) = \varepsilon / \alpha\beta$ (AP-reduction)

## APX-complete problems so far:

- Max E3SAT-E5
- Max 3SAT-3
- Independent set
- Vertex cover
- Dominating set

} bounded degree

## Max 2SAT: [Papadimitriou & Yannakakis – JCSS 1991]

- L-reduction from Independent set, bounded deg.
  - $v \Rightarrow v$ — $\forall$ vertex $v$
  - $v$ — $w$ $\Rightarrow \neg v \vee \neg w$ — $\forall$ edge $\{v, w\}$
  - never worth violating edge constraint: could violate either vertex at same cost
  - $\Rightarrow$ solution gives an indep. set
  - $\Rightarrow \mathrm{OPT}_{2SAT} = \underbrace{\mathrm{OPT}_{IS}}_{\Theta(|V|)} + \underbrace{\#\text{edges}}_{\Theta(|V|)}$ — bounded degree

## Max E2SAT-E3 → [Berman & Karpinski – ICALP 1999]

## Max NAE 3SAT: [Papadimitriou & Yannakakis – JCSS 1991]

- strict-reduction from Max 2SAT
  - $x \vee y \Rightarrow \mathrm{NAE}(x, y, a)$
    - ↑ same in all clauses
- by flipping, can assume $a = \emptyset$
- score = # $(x,y)$s where $x$ or $y = 1$

## Max cut: [Papadimitriou & Yannakakis — JCSS 1991]

- = max positive 1-in-2SAT
- = max positive XOR-SAT
- L-reduction from Max NAE 3SAT:
  - clause gadget: 2 points if satisfied, 0 else
  - variable gadget: never hurts to put $x_i$ & $\bar{x}_i$ in opposite sides

$\Rightarrow OPT_{cut} = 2 \cdot (\sum_i \#\text{ occurrences of } x_i$ → ≤ 3·# clauses
$+ \#\text{ satisfied clauses})$
$= \Theta(OPT_{NAE})$ → ≥ ½ # clauses

- degree-3 possible [Berman & Karpinski — ICALP 1999]

≤ max E2-LIN-$\mathbb{Z}_2$-3

E → =2 literals/eqn.

LIN-$\mathbb{Z}_2$ → linear eqns. over $\mathbb{Z}_2$

3 → 3 eqns./variable

# Max/min CSP / Ones:

\# clauses — \#true variables

[Khanna, Sudan, Trevisan, Williamson — SICOMP 2001]

- analog to Schaefer Dichotomy
- given allowable clause functions
- instance can be weighted or not
- e.g.: MaxE2SAT = Max CSP($x_1 \vee x_2$, $\bar{x}_1 \vee x_2$, $x_1 \vee \bar{x}_2$, $\bar{x}_1 \vee \bar{x}_2$)
  Max Cut = Max CSP($x_1$ XOR $x_2$)
  Max Clique = Max Ones ($x_1$ NAND $x_2$)
- Max CSP
  - $\in$ PO if setting all vars. false or all vars. true satisfies all clause types
  - $\in$ PO if all clauses in DNF have 2 terms, one all positive & one all negative
  - APX-complete otherwise
- Max Ones:
  - $\in$ PO if setting all vars. true satisfies all
  - $\in$ PO if CNF of Dual-Horn subclauses ($\le 1$ negated)
  - $\in$ PO if $\le$2-X(N)OR-SAT: linear eqns., 2 terms, over $\mathbb{Z}_2$
  - APX-complete if $\le$X(N)OR-SAT (not 2-)
  - Poly-APX-complete if CNF of Horn subclauses
  - Poly-APX-complete if 2CNF
  - Poly-APX-complete if setting all or all but one variable false satisfies each constraint
  - 0 vs. >0 NP-hard if setting all vars. false satisfies (& not previous case)
  - feasibility NP-hard if none of above

- Min CSP:
  - $\in$ PO if setting all vars. false or all vars. true satisfies all clause types
  - $\in$ PO if all clauses in DNF have 2 terms, one all positive & one all negative
  - APX-complete if $\underbrace{\text{OR}(O(1) \text{ variables})}_{O(1)\text{-hitting set}}$, $\underbrace{\neg x_1 \vee x_2}_{\text{implication}}$
  - $\underbrace{\text{Min UnCut}}_{\text{Min CSP(XOR)}}$-complete if $\leq$ 2-X(N)OR-SAT
    Min CSP(XOR) — APX-hard & $O(\log n)$-approx.
  - $\underbrace{\text{Min 2CNF-Deletion}}$-complete if 2CNF
    Min CSP(OR, NAND) ~ APX-hard & $O(\log n \log\log n)$-apx.
  - $\underbrace{\text{Nearest Codeword}}$-complete if $\leq$ X(N)OR-SAT (not 2-)
    Min CSP($x_1 \oplus x_2 \oplus x_3$, $\overline{x_1 \oplus x_2 \oplus x_3}$) — $\Omega(2^{\log^{1-\varepsilon} n})$-inapprox.
  - $\underbrace{\text{Min Horn Deletion}}$-complete if Horn or Dual-Horn
    Min CSP($\bar{x}_1 \vee x_2 \vee x_3$) — $\Omega(2^{\log^{1-\varepsilon} n})$-inapprox., $\in$ Poly-APX
  - 0 vs. >0 is NP-complete otherwise
- Min Ones:
  - $\in$ PO if setting all vars. false satisfies all
  - $\in$ PO if CNF of Horn subclauses ($\leq$ 1 positive)
  - $\in$ PO if $\leq$ 2-X(N)OR-SAT
  - APX-complete if 2CNF
  - APX-complete if $O(1)$ hitting set + implication
  - Nearest Codeword-complete if $\leq$ X(N)OR-SAT (not 2-)
  - Min Horn Deletion-complete if CNF of Dual-Horn
  - Poly-APX-complete if all vars. true satisfies — if weighted: hard to approximate by any factor
  - feasibility NP-hard otherwise

# Another APX-completeness series:

**Max. independent set** in 3-regular 3-edge-colorable graphs [Chlebík & Chlebíková – TCS 2006]

**Max. 3DM-E2:**

- given triples $\subseteq A \times B \times C$ (disjoint)
- solution = subset of triples not repeating any item $\in A \dot\cup B \dot\cup C$
- each item appears exactly twice
- strict-reduction from previous problem:
  - edge color classes → A, B, C
  - vertex → triple

**Max. edge matching puzzles:** [Antoniadis & Lingas – SOFSEM 2010]

- goal: maximize # matching edges
- $\in$ APX (max. matching gives $\geq n/8$ matches)
- L-reduction from previous problem
  - $2 \times n$

$\Theta(1)$-approximable ($\in$ APX $\setminus$ PTAS) but not APX-complete: [Crescenzi, Kann, Silvestri, Trevisan - SICOMP 1999]

(unless polynomial hierarchy collapses)
PH

Bin packing: given $n$ numbers & bin size $B$, min. # bins to store the numbers

- has asymptotic PTAS (+1 additive error)

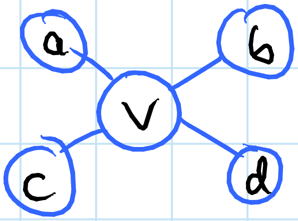
Min. max.-degree spanning tree

Min. edge coloring

## Log-APX-complete: (A-reductions: $y'$ c-approx. $\Rightarrow$ $y$ O(c)-approx.)

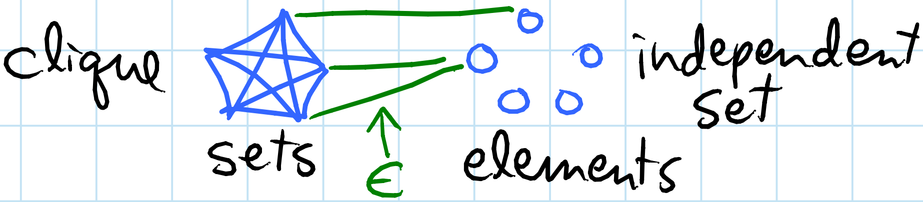
- set cover
- dominating set [Escoffier & Paschos — TCS 2006]

- strict-reduction from dom. set to set cover:




$\Rightarrow$ $S_v = \{v, a, b, c, d\}$

- strict-reduction from set cover to dom. set:




  - never need to choose element: take a set ⇒

## Token reconfiguration: [Calinescu, Dumitrescu, Pach — LATIN 2006]

- given initial & goal token placements
- move = slide pebble along empty path
- goal: min. # moves
- APX-hard for unlabeled & labeled tokens
  - L-reductions from Set Cover
- 3-approx. for unlabeled

- motivation: $15 = n^2 - 1$ puzzle
  - NP-hard & $\in$ APX [Ratner & Warmuth 1990]

Poly-APX-complete: max. independent set
& max clique (complement)

(PTAS-reductions) [Bazgan, Escoffier, Paschos - TCS 2005]

Exp-APX-complete: nonmetric Traveling salesman
$\hookrightarrow 2^{n^{O(1)}}$ [Escoffier & Paschos - TCS 2006]

## NPO-complete: THE HARDEST! (AP-reductions)

[Crescenzi, Kann, Silvestri, Trevisan – SICOMP 1999]

### Max./min. weighted SAT (AKA "ones")

- given CNF formula
- given nonneg. weight $w_i$ of each var. $x_i$
- solution = satisfying assignment (NP-hard!)
- cost = $\sum_i w_i x_i$ (can max with 1)

### Max/min 0-1 linear programming:

- given integer matrix $A$, vectors $b$ & $c$
- max/min $c \cdot x$ → 0-1 vector
  subject to $Ax \geq b$

## NPO PB-complete: above with integer → $\{0,1\}$ (or poly. bounded integer)

- $n^{1-\varepsilon}$-inapproximable even with trivial solutions [Jonsson – IPL 1998]
- min. independent dominating set ← [Kann – NJC 1994]
- shortest computation in nondet. Turing machine
- longest induced path ← [Berman & Schnitger – I&C 1992]
- longest path with forbidden pairs