Recall: packing of n squares without rotation into a square is strongly NP-complete \[\text{[L2]}\]

**Edge-unfolding polyhedra:** given a polyhedron, cut along edges to unfold flat without overlap

- not always possible \([\text{Biedl et al.} \& \text{Bern et al. 1998}]\)
- strongly NP-hard \([\text{Abel} \& \text{Demaine 2011}]\)
even for orthogonal polyhedra topologically sphere

**Reduction from Square Packing:**
- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out
  \[\Rightarrow\] squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface
  \[\Rightarrow\] can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation
Snake cube puzzle: AKA Cubra circa 1990
- given chain of unit cubes each with specified "turn angle" of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)
- NP-hard

[Abel, Demaine, Demaine, Eisenstat, Lynch, Scharf 2012]

Reduction from 3-Partition:
- infrastructure:
  - fill cube to leave \(x \times y \times z\) box
  - fill box to leave "hub & slots" shape
  - each hub is \(8 \times t \times \text{huge}\)
- \(a_i\) gadget: \(8a_i\) must go in 1 hub

- 8 to avoid coming back to same \(4 \times 4 \times 4\) voxel
- connected together by zig-zag gadget
- zig-zag is universal:
  - \(2 \times 2 \times 2\) can turn/go straight
  \(\Rightarrow\) fill Hamiltonian shapes scaled \(2 \times\)
  - \(2 \times 2 \times 2\) refinement makes any shape Hamilton.
  \(\Rightarrow\) \(4 \times 4 \times 4\) refinement makes fillable by zig-zag
- parity issue: snake alternates in cell parity
- claim: can start & end at any faces of opposite parity
Disk packing: pack n given disks into given shape

- motivation: computational origami design
  (tree method — see Lang)
- strongly NP-hard [Demaine, Fekete, Lang—OSME 2010]

Reduction from 3-Partition:
- infrastructure:
  - build \( n/3 \) symmetric \( \theta \) pockets
  - equilateral \( \Delta \): forced packing
  - square target: forced packing
    + repeated subdivision with forced packings
    + fill all other pockets by repeatedly adding maximal disks, until small enough (depth \( \approx \log n \))
- triple gadget: (in symmetric pocket)
  - scale \( a_i \)'s & \( t \) so that \( t = 1 \)
  - shrink center disk by \( -1/N \)
  - shrink \( a_i \) disk by \( -1/N^2 \), \( a_{big} \)
  - grow it by \( +a_i/N \)
- key property: disks fit \( \iff a_i + a_j + a_k \leq t \)
  (proof by geometry + Taylor series)
Clickomania: [Schuessler ~2000?]
- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
- remaining squares fall within each column
- empty columns disappear

- polynomial for one row or column
- reduces to CFG parsing
- NP-hard for
- 2 columns & 5 colors
- 5 columns & 3 colors
- OPEN: 2 rows? 2 colors?

Reduction from 3-Partition:
- left column mostly checkerboard except middle & interspersed red □s to measure t’s
- collapses ⇔ red □s removed
- right column has $a_i$ groups + red squares on top
- details: spacing out groups & reds while still getting alignment

[Redrawn] necessary: encoding in unary

$S \rightarrow \Delta | SS |$ $c_i Sc_i 1$
$c_i Sc_i Sc_i$
Tetris: [Alexey Pazhitnov 1985]
- rectangular board
- **tetromino** blocks come one at a time
  - 4 unit squares joined edge-to-edge
  
- can rotate block as it falls from sky
- filled lines disappear
- stack to sky \(\Rightarrow\) die

- **perfect information** version:
  - know entire sequence of pieces to come
  - initial board position given

- NP-complete to [Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2003]
  - survive
  - approximate \# lines/Tetrises/time until death
    up to a factor of \(n^{1-\varepsilon}\)

**Reduction from 3-Partition:**
\(\Rightarrow\) necessary: encoding in unary
- initial board = \(\frac{n}{3}\) buckets of "depth" \(\varepsilon\)
- \(a_i\) encoded as \(\square, (\square, \square, \square)^{a_i}, \square, \square\)
  - claim: entire gadget must go in one bucket
- finale = \((\square)^{n/3}, \square,(\square)^{\frac{5}{4}\varepsilon+4}\)

**Reachability:** essentially all Tetris boards are reachable from initial state
[Hoogeboom & Kosters - IJIGS 2005]
OPEN:
- initially empty board
- $O(1)$ rows or columns
- restricted piece sets (e.g. 
- no last-minute slides
- 2-player: PSPACE-complete?
- online Tetris?

1-planarity: draw a given graph in the plane such that each edge crosses $\leq 1$ other

- NP-complete [Ringel 1985]

Reduction from 3-Partition:
- uncrossable edge gadget:

  (denoted by thick edge)

- double wheel gadget:
  - unique embedding
  - one for $A$
  - one for triples
  - separate triples with thick edges every $t$ hours around triples gadget

- $a_i$-gadget:
  A center triples center
GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection
\[ \rightarrow \text{NP-complete from 3-Partition} \]

Ruler folding:
- given carpenter's ruler with lengths \( a_1, a_2, \ldots, a_n \)
- goal: fold to fit in 1D box of length \( L \)

- weakly \text{NP-complete} [Hopcroft, Joseph, Whitesides - 1985]
- pseudopolynomial (like 2-Partition)

Reduction from (2-)Partition:
- idea: Partition solvable \( \iff \) can assign signs to \( a_i \)'s such that \( \sum_i \pm a_i = 0 \)
- folding flips sign; unfolding leaves sign
  \[ \Rightarrow \text{can fold ends together} \iff \text{Partition solvable} \]
- construction: \( 2B, B, a_1, a_2, \ldots, a_n, B, 2B \)
  \[ \Rightarrow \sum_i a_i \]

\[ \Rightarrow \text{2B's will be aligned & fit inside length-2B box} \]
\[ \Rightarrow \text{can fold ends together} \iff \text{Partition solvable} \]
Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard [Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena - 2000]

for orthogonal paper & orthogonal creases
or square paper & 45° orthog. creases

Reduction from Partition:
- Similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check if ruler folded? 5 if not

- force square paper into orthogonal shape:

OPEN: strongly NP-hard? pseudopolynomial?