**PPAD:** definition later — start with motivation

**Motivation 1:** Economic Game Theory

**Game:**
- $n$ players $1, 2, ..., n$
- for each player $p$: set $S_p$ of strategies
- payoff for each player $p$:
  $u_p : S_1 \times S_2 \times \cdots \times S_n \to \mathbb{R}$
- e.g. Penalty Shot Game

**Nash Equilibrium** = locally optimal product distribution of strategies for the players such that no one player can (by changing just their strategy) improve their expected payoff.

i.e. $x_1, x_2, ..., x_n$ such that $\forall p$:

$E[u_p(x_1, ..., x_p, ..., x_n)] \geq E[u_p(x_1, ..., x'_p, ..., x_n)]$

$\forall x'_p \in \text{D}(S_p)$

- e.g. 1/2 - 1/2 strategies in Penalty Shot Game
- exist in 2-player zero-sum games \[\text{[von Neumann 1928]}\]
- via linear programming
- exist in $n$-player games \[\text{[Nash 1950]}\]
- still no poly-time algorithm to find them
Motivation 2: Brouwer's Fixed-Point Theorem
for any convex, closed, bounded set $S$, any continuous map $f: S \to S$ has a fixed point $p \in S$: $f(p) = p$ [Brouwer 1910]

Nash's proof via Brouwer's Theorem
- $f: [0,1]^n \to [0,1]^n$ is essentially a vector field indicating how each player can improve their mixed strategy (distribution)
- fixed point of $f =$ Nash equilibrium

Motivation 3: Sperner's Lemma
- square grid graph + backslash diagonals
- assign vertices 3 colors

2D version: if boundary is legally colored then there are an odd number ($\geq 1$) of trichromatic $\Delta$

d-dimensional version too (not covered here)
Proof of Brouwer via Sperner:
- for all $\varepsilon$, show approximate fixed point: $|f(x) - x| < \varepsilon$ via Sperner's Lemma
- color points according to direction of $f(x) - x$ (which of 3 boundaries)
- use compactness to take limit $\varepsilon \to 0$ (may not preserve oddness of solution count)

Computational version of Sperner:
- grid of size $2^n \times 2^n$
- internal vertex colors given by circuit $C$
- boundary in canonical legal coloring
- goal: find trichromatic $\Delta$

Computational version of Nash:
- given $n$, enumeration of strategy set $S_p$ & utility function $u_p : S \to \mathbb{R}$ of every player $p$
- goal: $\varepsilon$-Nash equilibrium
  - expected payoff can't improve by more than $+\varepsilon$
  - avoids representation issue for irrational equilibria (required for e.g. $n=3$ game)
Search problem defined by relation $R \subseteq \{0, 1\}^* \times \{0, 1\}^*$
where $(x, y) \in R$ means $y$ is solution to $x$

Total if $\forall x \exists y : (x, y) \in R$ i.e., always $\exists \geq 1$ solution
- e.g., Sperner & Nash & Brouwer

$\overline{FNP} = \{NP \text{ search problems}\}$
$\overline{FNP}$-complete $\in \overline{FNP}$ & $\exists$ one-call (Karp) reduction from every problem $\in FNP$
- impossible for total problems reducing from non-total problem e.g., SAT

Complexity theory for total problems: (TFNP)
- identify combinatorial argument for existence proof
- define complexity class
- check tightness via completeness result

Proof of Sperner’s Lemma:
- add artificial trichromatic $\Delta$ at boundary
- define directed walk from that $\Delta$:
  keep crossing bichromatic edges with same 2 colors with same orientation (else find trichromatic $\Delta$)
- can’t exit square by valid boundary coloring
- can’t form a cycle $\Rightarrow$ (uncolorable)
- for odd number theorem: can walk from every other trichromatic $\Delta$ to another $\Rightarrow$ even # except for one from boundary
Directed parity argument:
- vertices of graph represent $\Delta$s
- all vertices have in & out degrees $\leq 1$
  $\Rightarrow$ graph = disjoint union of directed paths, cycles, & isolated vertices
- degree-1 vertex = trichromatic $\Delta$
- degree-2 vertex = walkable (2 bichromatic edges with right orientation)
- degree-0 vertex = rest

Nonconstructive step: if there's an unbalanced vertex then there's another in-deg. $\neq$ out-deg.

End of the Line:
- each vertex $v$ has candidate incoming & outgoing edge $P(v)$ & $N(v)$
  - given as circuit: $V \xrightarrow{\bigcirc} size 2^n$
  - actual edge $(v,w) \iff$ both ends agree:
    $N(v) = w \land P(w) = v$
- goal: if $O^n$ is unbalanced, find another unbalanced node checkable in $O(n)$ time (4 circuit evaluations)
- $\text{EFNP}$: certificate = another unbalanced node

$\text{PPAD} = \{\text{search problems } \in \text{FNP reducible to}\}$

[End of the Line]

[Papadimitriou 1994]
So: \( \text{Nash} \rightarrow \text{Brouwer} \rightarrow \text{Sperner} \rightarrow \text{PPAD} \)

In fact: \( \text{Nash} \leftarrow \text{Brouwer} \leftarrow \text{Sperner} \leftarrow \text{PPAD} \)

i.e.: \( \text{Nash, Brouwer, Sperner are PPAD-complete} \)

\( \text{[Papadimitriou 1994]} \)

\( \text{[Daskalakis, Goldberg, Papadimitriou 2006]} \)

- even for 2-player Nash \( \text{[Chen & Deng 2006]} \)

Proof sketch: generic PPAD

\( \rightarrow \) embed graph in \([0,1]^3\)

\( \rightarrow \) 3D Sperner

\( \rightarrow \) Arithmetic Circuit SAT

\( \rightarrow \) Nash
Arithmetic Circuit SAT:
- **Input**: variable nodes $x_1, \ldots, x_n$ \leftarrow in degree 1
gate nodes $\oplus \rightarrow +$ etc. \leftarrow in degree $c \in \{0, 1, 2\}$
cycles allowed
arbitrary out degrees
- **Goal**: assignment of values $c[0, 1]$ to $x_1, \ldots, x_n$
satisfying all gate constraints:
  - $x \xrightarrow{\oplus} y \Rightarrow y = x$
  - $x \xrightarrow{+} y \Rightarrow z = x + y$
  - $c \xrightarrow{\times} x \Rightarrow x = c$ \text{ for constant $c$}
  - $x \xrightarrow{\times c} y \Rightarrow y = c \cdot x$ \text{ ce $[0, 1]$}
  - $x \xrightarrow{> \circ} y \Rightarrow z = \begin{cases} 0 & \text{if } x < y \\ 1 & \text{if } x > y \\ \text{arbitrary if } x = y \end{cases}$ \text{ weird but necessary}
- **Total**: always a satisfying assignment \text{ not obvious}
- **PPAD-Complete**

- improvement from exponential noise tolerance
  \rightarrow polynomial noise tolerance \text{[Chen, Deng, Teng 2006]}
  \text{Approximate Arith. Circuit SAT}