3SUM: [Gajentaan & Overmars - CGTA 1995]
given $n$ integers, do any 3 sum to 0?
  (allowing same integer to be chosen > once)
- conjecture: no $O(n^{2-\epsilon})$ algorithm
  "truly subquadratic"
- $O(n^2)$ randomized algorithm:
  - compute all pairwise sums
  - look in hash table of all negations
- $O(n^2)$ deterministic algorithm:
  - presort integers
  - for each target sum (negated integer):
    - advance left pointer right if too small
    - advance right pointer left if too big
- $O(n + u \log u)$ via FFT if integers $\in [-u,u]$
- $O(n^2/\left(\log n\right)^3)$ randomized in word RAM
  [Baran, Demaine, Pătraşcu- Alg. 2008]
- $O(n^2/\left(\log n\right)^2)$ det.: $O(n^2/(\log n)^{2/3})$ rand. in real RAM
  $O(n^{1.5}/\sqrt{\log n})$ in decision tree model
  [Grønlund & Pettie- FOCS 2014]
**k-SUM**: given \( n \) integers, do any \( k \) sum to 0?
- \( O(n^{\frac{k}{2} + \epsilon}) \) randomized algorithm
- conjecture: no \( O(n^{\frac{k}{2} - \epsilon}) \) algorithm
- \( \text{NP-complete} \) for \( k \) an input (\( \cong \text{Partition} \))
- \( \text{W}[1]-\text{hard} \) w.r.t. \( k \) (but quadratic parameter blowup from Clique \( \Rightarrow n^{o(\frac{n^2}{k})} \) lower bound)
- ETH \( \Rightarrow \) no \( n^{o(k)} \) algorithm for k-SUM
  \[ \leq n^{0.99} \] (Pătraşcu & Williams-SODA 2010)

**3SUM-hard** = \( O(n^{3 - \epsilon}) \) algorithm \( \Rightarrow \) one for 3SUM

**3SUM reduction** = \( \Theta(1) \)-call reduction on \( n' = O(n) \) running in \( O(n^{3 - \epsilon}) \) time
- A 3SUM-hard (e.g. 3SUM) \( \Rightarrow \) B 3SUM-hard

Base 3SUM-hard problems: (all equivalent)
- 3SUM with \( u = n^3 \) via hashing [Pătraşcu-STOC 2010]
- Distinct 3SUM: \( \exists 3 \) distinct integers summing to 0?
  - reduction from 3SUM: also check for doubled/tripled ints
  - reverse reduction?? [Mikhail Rudoy, today]
- 3SUM': given sets \( A, B, C \) of \( n \) integers \( \exists a \in A, b \in B, c \in C \) such that \( a + b = c \)?
  - reduction from 3SUM: \( A = B = S, C = -S \) (or \( a + b + c = 0 \))
  - also reduction in reverse direction [Gajentaan & Overmars-CGTA 1995]
- **GeomBase**: given n points in 2D with \( y \in \{0, 1, 2\} \), do nonhorizontal line hitting 3 points?

- **reduction from/to 3SUM**:
  - \( axA \leftrightarrow (a, 0) \quad a + b = c \)
  - \( bxB \leftrightarrow (b, 2) \quad c/2 = a + b \)
  - \( cxC \leftrightarrow (c/2, 1) \)

  \[ \text{[Gajentaan & Overmars - CGTA 1995]} \]

**More 3SUM-hard problems:**

- also solvable in \( O(n^2) \) time

- **3 points on a line**: given n points in the plane, are any 3 collinear?

  - **reduction from Distinct 3SUM**
  - \( x \in S \rightarrow (x, x^3) \)

**Point on 3 lines**: given n lines in the plane, do any 3 meet at a point?

- **projective plane dual of 3 points on line**:
  - \((a, b) \leftrightarrow ax+by+1 = 0\)
  - \((\text{lines } ax+by = 0 \text{ passing through origin map to points @ infinity ~ avoid these})\)
  - preserves point/line incidence

**d-D versions**: \((d+1)-SUM\) hard
Separator: given \( n \) segments, is there a line splitting them into 2 nonempty groups?
- reduction from GeomBase
- if allow half-infinite segments, can all be horizontal (Sep.1)
- else horizontal & vertical segments (Sep.2)

Strips cover box: does union of \( n \) strips cover a given axis-aligned rectangle?
- reduction from GeomBase
- start from Separator 1 reduction rotated 90°
- dualize: \((m, b) \rightarrow y = mx + b\)
  - vertical segment \( \rightarrow \) strip
  - half-infinite segment \( \rightarrow \) half plane
- rectangle = bounding box of hexagonal hole in union of 6 half-planes
- restrict half planes to this rectangle
  \( \rightarrow \) 6 more strips
- uncovered point in dual
  = line in primal not hitting any segments
Triangles cover triangle:
- reduction from previous problem
- convert box $\rightarrow$ triangle with $O(1)$ strips
- split strips into 2 large $\Delta$s
- can assume $n$ triangles $\leq$ big triangle;
  - replace each triangle with intersection
  - triangulate resulting $O(1)$-gons

Hole in union: does union of $n$ triangles have a hole?
- reduction from previous problem ($\leq$ version)
- add thin $\Delta$s covering edges of big $\Delta$
- hole $\Leftrightarrow$ not covered
- reduction in reverse direction also possible

Triangle measure: area of union of $n$ triangles
- reduction from Triangles cover triangle ($\leq$)
- $\text{area(union)} = \text{area(big $\Delta$)} \Leftrightarrow$ covered

Point covering: is there a $k$-way intersection between $n$ given half planes?
- reduction from Strips cover box
- strip $\rightarrow$ complement as 2 half planes
- rectangle $\rightarrow$ 4 half planes whose int. = rect.
- $k = n + 4$ (outside $n$ strips, inside rectangle)
Visibility between segments:
given n horizontal segments, is there a point on segment 1 that can see a point on segment 2 (unobstructed by segments)
- reduction from GeomBase like Separator 1

Visible triangle: given n horizontal triangles in 3D
can a given point see a point on triangle 1?
- reduction from Triangles cover triangle (view from infinity)
- reduction in reverse direction too

Planar motion planning: can you move segment robot through
horizonal & vertical segment obstacles?
- reduction from GeomBase (like Separator 1)

3D motion planning: can you translate vertical segment robot through horizontal Δ obstacles?
- reduction from Triangles cover triangle
- separate Δs slightly in z, in middle of cage
- goal: get from top half to bottom half of cage
- O(n² log n) algorithm
Fixed-angle chains: [Soss, Erickson, Overmars 2002]
which edge-spin operations cause collisions in a given fixed-angle chain?
- reduction from 3SUM'
- subtract $2M$ from each $aeA \rightarrow A'$
- add $2M$ to each $ceC \rightarrow C'$
  \[
  \max \text{ abs } (A \cup B \cup C)
  \]
- best algorithm: $O(n^3)$ [Soss & Toussaint 2001]

Nonquadratic lower bounds: [Pătraşcu - STOC 2010]
- finding $\Delta$ of prescribed weight in a weighted graph in $O(E^{1.5-\varepsilon})$ time is 3SUM-hard (as hard as $O(n^{2-\varepsilon})$ for 3SUM)
- finding $|E|/\Delta$s in $O(E^{4/3-\varepsilon})$ time is 3SUM-hard
Conjectured cubic graph problems: (weighted)

**Diameter**: \( \max_{v,w} S(v,w) \) in undirected graph

- **conjecture**: no \( O(V^{3-\epsilon}) \)-time algorithm
- no \( (3/2-\epsilon) \)-approx. in \( O(E^{2-\epsilon}) \) time, even unweighted, assuming Strong ETH
  [Raditya & Vassilevska Williams - T.ALG 2012]

- subcubic reduces to: \( O(n^{3-\epsilon}) \)

**APSP** (All-Pairs Shortest Paths): \( S(v,w) \forall v,w \)

- \( O(V^3) \) via Floyd-Warshall algorithm (relax all edges \( |V| \) times)
- **conjecture**: no \( O(V^{3-\epsilon}) \)-time algorithm
- **APSP-hard**: no \( O(V^{3-\epsilon}) \) alg. assuming

**Negative \( \Delta \)**: is there a 3-cycle of negative weight?

- **APSP-hard** \( \sim \) actually equivalent
- equivalent to listing \( |V|^{0.99} \) negative \( \Delta S \)
- equivalent to testing \( \Delta \) inequality
  [Vassilevska Williams & Williams - FOCS 2010]

**Radius**: \( \min_v \max_w S(v,w) \) \[ Abboud, Grandoni, Vassilevska Williams - SODA 2015 \]

**Median**: \( \min \sum_w S(v,w) \)

- **APSP-hard** \( \sim \) actually equivalent (directed or undirected)