0-player games (simulations)
- polynomial # moves $\rightarrow$ P
- polynomial space $\rightarrow$ PSPACE
- infinite space $\rightarrow$ undecidable

Conway's Game of Life: [Conway 1970]
- cellular automata
- live cell lives $\Leftrightarrow$ exactly 2 or 3 live neighbors
- dead cell becomes live $\Leftrightarrow$ exactly 3 live neighbors
- PSPACE-complete in finite board
  - Turing machine simulation [Paul Rendell 2000]
    (pushdown automaton with 2 stacks)
- undecidable in infinite board (dead outside input)
- growing Turing machine [Paul Rendell 2000]
- 2-counter machine = Minsky machine $\leq$ exponential slowdown!
  [Conway, Berlekamp, Guy - Winning Ways 1982]
- wire, terminator, turn, delay
- shift: many offsets cause glider destruction
- AND & OR gates
- kickback $\rightarrow$ thinning $\rightarrow$ crossover
- split & NOT (complicated)
- counter registers, test, create, push/pull
- precise glider positioning away from guns
- self-destruction via boomerangs
Deterministic Constraint Logic: (DCL)
- edges can also be active or inactive
- vertex active if its active incoming edges' weight ≥ 2
- in each round:
  - reverse inactive edges pointing to active vertices
  & reverse active edges pointing to inactive vertices
- these are the new active edges

- PSPACE-complete even for planar AND/OR graphs
  - guarantee gadget inputs reverse at t = 0 (mod 4)
  - quantifier gadgets use new "switch" & degree-2 vertices to control timing
- CNF formula uses AND', OR', split' gadgets which take inputs & return acknowledgments (fixes timing & "blow-back")
- trick to guarantee first input of AND' activates before second (if they both do)
- remove degree-2 vertices
  - edge → 4-path & remove red-red vertices
  - remove blue-blue vertices
  - remove red-blue vertices (timing is OK)
- crossover gadget
Multiplayer games:
- typical question: given a game position, can next-player-to-move force a win?
- in worst case, other players collude against you, effectively acting as one player

2-player games:
- call players "white" & "black" (as in Chess, Go, ...)
- polynomial # moves $\rightarrow$ $\in$ PSPACE:
  - 1 move: 4 responses: 1 move: 4 responses: ... (rules & I win, in 3CNF) $\rightarrow$ Q3SAT

SAT games: [Schaefer - JCSS 1978]
- QSAT is a 2-player game: $G_w$(CNF)
  - player 1 chooses $x_1$, player 2 chooses $x_2$, ...
  - player 1 wins $\iff$ formula satisfied

- impartial games: (both players have same moves)
  - on turn, player sets any unassigned variable

- partizan games: (different moves for players)
  - white variables & black variables (50/50%)
  - on turn, player sets unassigned var. of same color

- default game: player 1 wins $\iff$ formula satisfied
- seek game: win if first to satisfy formula
- avoid game: lose if first to satisfy formula
- PSPACE-complete:
  - impartial game positive 11-SAT
  - impartial game positive 11-DNF SAT
  - partizan game CNF SAT
  - impartial/partizan avoid positive 2-DNF SAT
  - impartial/partizan seek positive 3-DNF SAT
  - impartial/partizan avoid positive CNF SAT
  - impartial/partizan seek positive CNF SAT

Kayles: \((\approx \text{indep. set})\) [Schaefer - JCSS 1978]
- (impartial) node Kayles:
  - on turn, player adds node to independent set
  - lose if can't move
- (partizan) bipartite node Kayles:
  - white vs. black nodes is the bipartition

Geography: (generalization of word game) \((\approx \text{longest path})\)
- given (directed) graph & start node for token
- on turn, player moves token along (directed) edge
- node geography: can't revisit nodes
  - directed PSPACE-complete [Lichtenstein & Sipser 1980]
  - undirected \(\in P\) [Fraenkel, Scheinerman, Ullman 1993]
- edge geography: can't revisit edges
  - directed PSPACE-complete [Schaefer - JCSS 1978]
  - undirected PSPACE-complete [Fraenkel, Scheinerman, Ullman - TCS 1993]
Reversi/Othello: \[<1883]\]
- move = \[ \begin{array}{cccc} 
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{\textcolor{red}{\text{0}}} & \mathbf{\textcolor{red}{\text{0}}} & \mathbf{\textcolor{red}{\text{0}}} & \mathbf{\textcolor{red}{\text{0}}} \\
\end{array} \] \[ \rightarrow \] \[ \begin{array}{cccc} 
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\textcolor{red}{\text{0}}} \\
\end{array} \]
- reverse in between 1 & 8 directions

- PSPACE-complete \[ \text{[Iwata & Kasai - TCS 1994]} \]
- polynomial # moves: move consumes board
- reduction from directed node geography in bipartite max-degree-3 graph
- rightward chains are threats by black: black takes \( x \), then \( x' \), then corner, then all of bottom territory \( \Rightarrow \) win
- white wins if black can't move
- degree-2 vertices: \[ \rightarrow \rightarrow \]
- degree-3 vertices: \[ \rightarrow \rightarrow \& \rightarrow \rightarrow \]
  
  if double visited then white or black wins

  by black or white chooses
A SIDE:

**Bounded NCL:** NP-complete
- each edge can be reversed only once
- NP-complete for planar constraint graphs with AND, SPLIT, OR, CHOICE vertices differ in initial edge orientations
  - can't expand without making new type of AND
- planar via crossover
- similar to proof of Constraint Graph Satisfaction

**Bounded 2-player Constraint Logic (2CL)**
- each edge is either white or black
- each edge can be reversed only once
- goal:
  - each player has target edge
  - player unable to move loses
- PSPACE-complete for planar constraint graphs with white AND, SPLIT, OR, CHOICE & VARIABLE vertex
- reduction from impartial game positive CNFSAT
- players take turns setting variables
- positive $\Rightarrow$ white wants true, black wants false
- black can't win (edge irreversible)
- white wins $\Leftrightarrow$ formula satisfied
- crossover gadget (only use of CHOICE)
- can make OR protected using free edge
  - no constraint at degree-1 end