NP search problem: \( \Leftrightarrow \) NP-relation
- goal: instance \( \rightarrow \) solution (any)
- for each instance, set of (valid/feasible) solutions
- can recognize instances & their solutions in P

- every NP problem \( \rightarrow \) NP search problem
  (for every choice of YES certificates \( \rightarrow \) solutions)

Counting version \#A of NP search problem A
- count number of solutions for given instance
- e.g. \#SAT: find \# satisfying assignments
  \#Shakashaka: find \# solutions to puzzle

\[ \text{#P} = \{ \text{#A} \mid \text{NP search problem A} \} \]  \[\text{Valiant-TCS 1979}\]
- problems solved by polynomial-time
  nondeterministic counting algorithms
- \( \Rightarrow \) makes guesses, at end says YES or NO
  (just like an NP algorithm)
  output = \#guess paths leading to YES

\#P-hard = as hard as all problems in \#P
via multical (Cook-style) reductions

\( \Rightarrow \) \#P unless \( P = \text{NP} \)
  \( \Rightarrow \) technically, \( \text{FP} = \) poly-time computable functions
Parsimonious reduction for NP search problems

- instance \( x \) of \( A \) \( \rightarrow \) instance \( x' \) of \( B \)
- computable in polynomial time (like NP reduction)
- \( \#A \) solutions to \( x = \#B \) solutions to \( y \)
- \( \Rightarrow \) decision problems (3 solution?) same answer
- \( \Rightarrow \) NP reduction too

- \( \#A \) is \( \#P \)-hard \( \Rightarrow \) \( \#B \) is \( \#P \)-hard

C-monious reduction: uniform scaling
- \( c \cdot \#A \) solutions to \( x = \#B \) solutions to \( y \)
- preserves \( \emptyset \) \( \Rightarrow \) NP reduction too
- \( \#A \) is \( \#P \)-hard \( \Rightarrow \) \( \#B \) is \( \#P \)-hard

\#P-complete SAT problems:

- \#3SAT
- planar \#3SAT [Hunte, Marathe, Radhakrishnan, Stearns 1998]
- planar monotone rectilinear \#3SAT
- planar positive rectilinear \#1-in-3SAT
- planar positive \#2SAT-3

- Schaefer-style dichotomy:
  - \( \#\text{SAT} \in \text{PFP} \Leftrightarrow \text{system of linear equations (mod 2)} \)
  - \( \#\text{SAT} \) \#P-complete otherwise

[Creignou & Hermann, I&c 2006]
see [Creignou, Khanna, Sudan, SIGACT 2001]
**Shakashaka**: parsimonious \(\Rightarrow\) \#P-hard

[Demaine, Okamoto, Uehara, Uno – CCCG 2013]

**Hamiltonian cycles:**
- old proofs not parsimonious [Lichtenstein] [Plesnik]
- parsimonious reduction from 3SAT to planar max-degree-3 Hamiltonian cycle
  [Sato – senior thesis 2002]
- nonplanar case solved earlier [Valiant 1974]

**Slitherlink**: parsimonious \(\Rightarrow\) \#P-hard
- here can't use grid graphs
  \(\Rightarrow\) optional vertex gadgets

[Yato 2000]
Determinant of \( n \times n \) matrix \( A = (a_{ij}) \):

\[
\text{Determinant} = \sum_{\text{permutation } \pi} (-1)^{\text{sign}(\pi)} \prod_{i=1}^{n} a_{i, \pi(i)}
\]

product of permutation matrix within \( A \)

Permanent:

\[
\text{Permanent} = \sum_{\text{permutation } \pi} \prod_{i=1}^{n} a_{i, \pi(i)}
\]

weight directed \( n \)-node graph \( w(i,j)=a_{ij} \):

\[
= \sum \text{product of edge weights } 1 \text{ cycle cover } 3 \text{ vertex-disjoint directed cycles hitting all vertices}
\]

- \#P-complete \cite{Valiant-TCS1979}
- c-monious reduction from \#3SAT
- weight-1 edges in variable & clause gadgets
- special weight matrix \( X \) in junctions
  - \( \text{perm } X = 0 \Rightarrow \) not alone in nonzero cycle cover
    \( \Rightarrow \) entered & exited by bigger cycle
  - \( \text{perm } (X - \text{row & col. 1}) = \text{perm } (X - \text{row & col. 4}) = 0 \)
    \( \Rightarrow \) can't enter & leave immediately
    \( \Rightarrow \) enter at one end (1 or 4), leave at other
  - \( \text{perm } (X - \text{rows & cols. 1 & 4}) = 0 \)
    \( \Rightarrow \) can't leave interior 2x2 separate
    \( \Rightarrow \) must be visited between enter & exit
  - \( \text{perm } (X - \text{row 1 - col. 4}) = \text{perm } (X - \text{row 4 - col. 1}) = 4 \)
    factor for each traversal
  - acts as forced edge in var. & clause gadgets
  - \( \text{perm} = 4^8 \cdot \# \text{clauses} \cdot \# \text{satisfying assignments} \)
Permanent $\bmod r$ also \#P-hard: [Valiant-TCS 1979]
- multicolor reduction from Permanent
- set $r=2, 3, 5, 7, 11, \ldots$ until product $> M^n \cdot n!$
  \[ \text{largest absolute entry in matrix} \leq \]
\[ \Rightarrow O(n \lg M + n \lg n) \text{ calls} \& \text{ max } r=O(\ln \ln n) \]
- use Chinese Remainder Theorem [Prime \# theorem]

$0/1$-permanent $\bmod r$:
- parsimonious reduction from permanent $\bmod r$
  \[ \Rightarrow \text{ all edge weights (effectively) nonnegative} \]
- replace weight-$k$ edge ($k>1$) with gadget with $k$ loops
  \[ \Rightarrow \text{ unique solution if original edge unused} \]
  \[ \Rightarrow \text{ exactly } k \text{ solutions if original edge used} \]
  \[ \text{(using exactly 1 loop)} \]

$0/1$-permanent:
- one-call reduction from $0/1$-permanent $\bmod r$
- call with same input
- return output $\bmod r$

= \# cycle covers in given directed graph
= \# perfect matchings in given bipartite graph
  \[ (V_1 = \text{rows}, V_2 = \text{columns}, (i,j) \in E \Rightarrow a_{ij} = 1) \]
\[ \overset{\text{balanced: } |V_1| = |V_2|}{} \]
Bipartite # maximal matchings: [Valiant - SICOMP 1977]
- one-call reduction from bipartite # perfect matchings
- replace each vertex with n copies (n=\(1\times l!\))
  & each edge with biclique \(K_{n,n}\)
  \(\Rightarrow\) old matching of size i
  \(\rightarrow (n!)^i\) distinct matchings of size ni
  (& preserves maximality)
- # maximal matchings
  \(= \sum_{i=0}^{n^2} \binom{\text{# orig. maximal matchings size i}}{i} \cdot (n!)^i\)
  \(\leq \left(\frac{n}{2}\right)!\), e.g. \(k\times n/2, n/2\)
  \(\Rightarrow\) can extract # perfect matchings (i=\(n/2\))

Bipartite # matchings: [Valiant - SICOMP 1977]
- multicall reduction from bipartite # perfect matchings
- \(G \rightarrow G_k\): for each vertex: add k adjacent leaves
- \(M_r\) matchings of size \(n/2-r\) in \(G\)
  contained in \(M_r (k+1)^r\) matchings in \(G_k\)
  \(\Rightarrow\) # matchings in \(G_k = \sum_{r=0}^{n/2} M_r (k+1)^r\)
- evaluate this polynomial for \(k=1, 2, \ldots, n/2+1\)
  \(\Rightarrow\) can extract coefficients \(M_0, M_1, \ldots\)
- \(M_0 = \) desired # perfect matchings in \(G\)
Positive #2SAT
\[\text{= \# vertex covers} \quad \Rightarrow \text{\# cliques in complement graph}\]

- parsimonious reduction from bipartite \# matchings
- edge \(\rightarrow\) variable: true = not in the matching
- 2 incident edges \(e \& f\) \(\Rightarrow\) clause \(e \lor f\)
  \(\Rightarrow\) satisfying assignment = matching

# Minimal Vertex Covers
\[\text{[Valiant - SICOMP 1977]}\]
\[\text{= \# maximal cliques in complement graph} \]
\[\text{= \# minimal truth settings for positive 2SAT}\]

- parsimonious reduction from bipartite
  \# maximal matchings, as above
- minimal satisfying assignment = maximal matching
  \[|E_i| - i \text{ true variables} \Leftrightarrow \text{size } i\]

3-regular bipartite planar \#Vertex Cover
\[\text{= planar positive 2SAT-3} \]
where each clause has 1 red & 1 blue variable
- \#P-complete \[\text{[Xia, Zhang, Zhao - TCS 2007]}\]

(2,3)-regular bipartite \# Perfect Matchings
- \#P-complete \[\text{[Xia, Zhang, Zhao - TCS 2007]}\]

(note: decision versions easy)
Another Solution Problem (ASP) [Ueda & Nagao - TR 1996]
- for NP search problem A:
  ASP A: given one solution, is there another?
- useful in puzzle design: want unique solution
- e.g. ASP k-coloring ∈ P \ (rotate colors)
  & ASP 3-regular Hamiltonian cycle ∈ P
  (always another solution)

ASP reduction: parsimonious reduction \( A \rightarrow B \)
& poly-time bijection between solutions \( A(x) \)
& solutions \( B(x') \)
- induces every parsimonious reduction we’ve seen
  \( \Rightarrow \) ASP \( A \rightarrow \) ASP \( B \) via NP reduction
  (can map given solution to \( A \rightarrow \) sol. to \( B \))
- ASP \( B \in P \Rightarrow \) ASP \( A \in P \)
- ASP \( A \) NP-hard \( \Rightarrow \) ASP \( B \) NP-hard

ASP-hard = ASP reducible from every NP search prob.
  \( \Rightarrow \) NP-hard

ASP-complete = ASP-hard NP search problem
- includes planar 3SATs & Hamiltonicity today,
  Shakashaka, Slitherlink
- not e-monius reductions: 2SAT, matchings, permanent