Exponential Time Hypothesis (ETH):
- there is no $2^{o(n)}$-time algorithm for 3SAT on # variables
  [Impagliazzo & Paturi - CCC 1999]
- current best algorithm is $1.30704^n$ [Hertli 2011]
  $\rightarrow$ # clauses
$\Leftrightarrow$ there is no $2^{o(m)}$-time algorithm for 3SAT
  [Sparsification Lemma - Impagliazzo, Paturi, Zane - JCSS 2001]
  - cf. $m = O(n^3)$
  - dense formula $\Rightarrow O(2^{\varepsilon n})$ sparse formulas $\notin \mathsf{E}$

Strong ETH: no $(2-\varepsilon)^n$-time alg. for CNF-SAT
(i.e. constant for $k$-SAT $\rightarrow 2$ as $k\to\infty$) [I&P]

3-coloring: (following lecture notes by Dániel Marx)
- recall NP-hardness reduction from 3SAT [L9]
  [Garey, Johnson, Stockmeyer - TCS 1976]
  - $n$ variables & $m$ clauses
  $\Rightarrow O(n+m)$ vertices & edges
- ETH $\Rightarrow$ no $2^{o(n)}$-time algorithm for 3-coloring
  graph where $|V| \& |E| = O(n)$
Size blowup of NP reduction: $|x| = n \xrightarrow{f} |x'| = b(n)$
- $T(n)$ alg. for $B \Rightarrow T(b(n))$ alg. for $A$
- no $2^{o(n)}$ for $A \Rightarrow$ no $2^{o(b^{-1}(n))}$ for $B$
- $b$ linear $\Rightarrow$ preserve "no $2^{o(n)}$-time alg."

**Vertex Cover:** ETH $\Rightarrow$ no $2^{o(n)}$-time algorithm for $|V| \& |E| = O(n)$
- e.g. L7/Lichtenstein 1982 reduction has linear blowup

**Dominating Set:** ditto
- e.g. L10/Papadimitriou & Yannakakis 1991 reduction from Vertex Cover

**Hamiltonicity:** ditto
- e.g. L7/Lichtenstein 1982 reduction or L8/Plesnik 1979 reduction $\Rightarrow$ max. deg. 3 has linear blowup
- not planar versions: maybe $\Theta(n^2)$ crossovers

**Independent Set:** ditto
- e.g. L10/Papadimitriou & Yannakakis 1991 reduction from 3SAT-3
  - need -3 to avoid quadratic # edges
Clique: ETH \implies \text{no } 2^{o(|V|)}\text{-time algorithm } (|E|=\Theta(|V|^3))

Planar 3SAT: L7/Lichtenstein 1982 reduction has quadratic blowup
- ETH \implies \text{no } 2^{o(|V|)} \& \text{no } 2^{o(|\neg \neg |V|)}\text{-time algorithm}
  \implies \#\text{vars.} \implies \#\text{clauses}

Planar 3-coloring, Vertex Cover, Dominating Set, Hamiltonicity, Independent Set: (NOT Clique)
- ETH \implies \text{no } 2^{o(|V|)}\text{-time algorithm}
  \text{for planar graphs with } n \text{ vertices (above reductions)} \implies O(n) \text{ edges}
  \text{[Cai & Juedes 2001]}

Parameterized consequences:
- no 2^{o(k)} \cdot n^{O(1)} \text{ algorithm for (k-) Vertex Cover, } k\text{-Path (Longest Path), Dominating Set, Independent Set, Clique}
  \text{not surprising - not even FPT if ETH holds}

- no 2^{o(|V|)} \cdot n^{O(1)} \text{ algorithm for Planar (no 3-coloring) Vertex Cover, Longest Path, Dom. Set, Ind. Set}
- 2^{O(|V|)} \cdot n^{O(1)} \text{ algorithms known}
  \text{[Alber, Bodlaender, Fernau, Kloks, Niedermeier 2002; Demaine, Fomin, Hajiaghayi, Thilikos - JACM 2005]}
**Stronger:** \( \text{ETH} \implies \text{no } f(k) n^{o(k)} \)-time algorithm for Clique/Indep. Set for any computable \( f \) [Chen, Huang, Kanj, Xia - JCSS 2006]

- reduction from 3-coloring
- split vertices into \( k \) groups of \( n/k \) vertices
- create graph with \( k \) groups of \( \leq 3^{n/k} \) vertices, one per valid 3-coloring of input graph
- connect 2 colorings if they are compatible

\( \implies k\text{-clique corresponds to 3-coloring} \)

- if k-Clique solvable in \( f(k) n^{k/s(k)} \)
  - then set \( k \) as large as possible such that \( f(k) \leq n \) & \( k^{k/s(k)} \leq n \)
  \( \implies k = k(n) \) is unbounded function (min of 2 inverses)

\( \implies \) running time on reduced graph

\[ f(k) \cdot (k^{3^{n/k}})^{k/s(k)} \leq n \cdot k^{k/s(k)} 3^{n/s(k)} \leq n^2 \cdot 3^{n/s(k(n))} = 2^{o(n)} \]

solution to 3-coloring contradicts ETH
Parameterized reduction: $x \rightarrow x'$ (recall L13)
- parameter preserving: $k'(x') \leq g(k(x))$
  \[ \text{parameter blowup} \]
- no $f(k) \cdot n^{o(k)}$ for $A \Rightarrow$ no $f'(k') \cdot n^{o(g^{-1}(k'))}$ for $B$
- e.g. no $f(k) \cdot n^{o(k)}$-time alg. for
  - Multicolored Clique/Indep. Set $\exists k' = k$
  - Dominating Set, Set Cover
  - Partial Vertex Cover (via better reduction)

Tool for parameterized complexity of planar problems:

**Grid Tiling** [Marx – FOCS 2007; ICALP 2012]
- given $k \times k$ grid, each cell $(i, j)$ with set $S_{ij}$ of 20 coordinates $\in \{1, 2, \ldots, n^2\}$
- goal: choose one $x_{ij} \in S_{ij}$ $\forall i, j$ such that
  - vertical neighbors agree in first coordinate
  - horizontal neighbors agree in second coordinate

- $W[1]$-hard & ETH $\Rightarrow$ no $f(k) \cdot n^{o(k)}$-time algorithm
- reduction from Clique, $V = \{v_1, v_2, \ldots, v_n\}$
  - $k' = k$, $n' = n$
  - $S_{ii} = \{v \mid v \in V \}$ $\forall i$
  - $S_{ij} = \{ (v, w) \in E \mid v \neq w \}$ $\forall i \neq j$
List coloring: given graph & list $L_v$ of valid colors for each vertex $v$, is there a coloring?

- NP-hard even for planar & $|L_v| \leq 3$ (3-coloring)
- parameterized by outerplanarity

\# times can remove all vertices from outside face

- $\in$ XP (bounded treewidth algorithm)
- $\text{W}[1]$-hard & ETH $\Rightarrow$ no $f(k) n^{o(k)}$ algorithm
- reduction from Grid Tiling
- colors $= \{1, 2, \ldots, n\}^2$ $\Rightarrow$ $S_{ij}$ set of colors
- $k \times k$ grid of vertices $u_{ij}$, list $= S_{ij}$
- between vertically adjacent vertices:
  vertex $v_{ijcd}$, list $= \{c, d\}$, connected to both
  $\forall$ colors $c,d$ not agreeing on first coord.
  $\Rightarrow$ vertical neighbors agree on first coord.
  (if one uses $c$, $v_{ijcd}$ used $d \Rightarrow d$ unavailable)
- between horizontally adjacent vertices:
  vertex $h_{ijcd}$, list $= \{c, d\}$, connected to both
  $\forall$ colors $c,d$ not agreeing on second coord.
  $\Rightarrow$ horizontal neighbors agree on second coord.
- by contrast: coloring is FPT w.r.t. outerplanarity (treewidth)
Grid tiling with \( \leq \):
- \( \text{first coord}(x_{i,j}) \leq \text{first coord}(x_{i+1,j}) \) (column)
- \( \text{second coord}(x_{i,j}) \leq \text{second coord}(x_{i,j+1}) \) (row)
- \( W[1] \)-hard & ETH \( \Rightarrow \) no \( f(k)n^{o(k)} \) algorithm
  - reduction from grid tiling
  - \( k' = 4k \)
  - \( 4 \times 4 \) gadgets

Scattered set: (d-independent set)
- given graph & numbers \( k \) & \( d \)
- find \( k \) vertices with pairwise distances \( \geq d \)

- \( d = 2 \) \( \Rightarrow \) Independent Set \( \Rightarrow W[1] \)-hard w.r.t. \( k \)
- planar graphs: FPT w.r.t. \( (k, d) \)
- planar graphs & \( d \) input:
  - \( n^{O(k)} \)-time algorithm
  - \( W[1] \)-hard w.r.t. \( k \) & ETH \( \Rightarrow \) no \( f(k)n^{o(\sqrt{k})} \) alg.
  - reduction from Grid Tiling with \( \leq \)
  - \( n \times n \) grid for grid cell \( (i, j) \)
  - color in \( S_{ij} \) \( \rightarrow \) length 100D path attached to corresponding grid node

- \( d = 301 n + 1 \)
- \( k' = k^2 \)
Unit-disk graphs:
- each vertex has coordinates in 2D (\(\mathbb{Q}^2\))
- edge \(\iff\) distance \(\leq 1\)

- independent set = radius-\(\frac{1}{2}\) disk packing with given centers
- \(n^{O(\sqrt{k})}\)-time algorithm [Alber & Fiala - J. Alg. 2004]
- \(W[1]\)-hard & no \(f(k)n^{o(\sqrt{k})}\)-time algorithm
  - reduction from Grid Tiling with \(\leq\)
  - \(k\times k\) unit grid of \(n\times n\) tiny grids of dots
    - with only \(S_{ij}\) dots present
  - \(k' = k^2\) (one per subgrid)

\(\Rightarrow\) no EPTAS unless \(FPT = W[1]\)

- \(ETH \Rightarrow\) no \(2^{(1/\varepsilon)O(1)}n^{O((1/\varepsilon)^{1-\delta})}\) \((1+\varepsilon)\)-approx. HS
  \(\Rightarrow\) \(n^{O(1/\varepsilon)}\)-time PTAS tight [Marx - FOCS 2007]