Hamiltonian cycle/tour/circuit: (default) = cycle visiting each vertex exactly once
Hamiltonian path = path visiting each vertex exactly once

History: Icosian Game [Sir William Rowan Hamilton 1857] [Astronomer Royal of Ireland]

Polynomial for (always yes)
- cubes of graphs
- planar 4-connected graphs [Karaganis 1968] [Tutte - Trans. AMS 1956]

NP-complete even:
(p) - given start & end vertices
- reduction from Ham. cycle: $s \rightarrow t \rightarrow s$
- for planar 3-regular 3-connected graphs with min. face degree 5 [Garey, Johnson, Tarjan 1976] [Krishnamoorthy 1975] [Chvátal 1976]
(c) - for bipartite graphs
(c) - for squares of graphs
(c) - Ham. cycle given a Ham. path [Papadimitriou & Steiglitz 1976]
NP-complete for planar directed max-degree-3 graphs
- reduction from 3SAT
- clause gadget
- XOR gadget
- crossover gadget

NP-complete for planar bipartite max-degree-3 graphs
- reduction from previous problem

Grid graph = vertices on square lattice + edges for all pairs at unit distance
- solid grid graphs: Hamiltonicity polynomial
- (with holes) Hamiltonicity NP-complete
- reduction from previous problem
  - parity-preserving grid embedding (via 3x scale)
  - edge (& turn) gadget "tentacles"
  - vertex gadget
  - 3 Ham. path from $p_i$ to $p_j$ visiting $e_1, e_2, e_3, e_4$
  - vertex-edge connections (parity dependent)

Euclidean TSP: NP-hard special case
Platform games with coins & time limit
Max-degree-3 grid graphs: [Papadimitriou & Vazirani, 1984]
- similar reduction from Planar 3SAT
- turn gadget → hole (but topologically same)
- vertex gadget = dumbbell
- vertex-edge connections:
  - degree-2 vertex: opposite ends of dumbbell
  - degree-3 vertex:
    - nonforced edges on opposite ends
    - forced edge on both ends via "fork"

Euclidean degree-3 MST: [Papadimitriou & Vazirani, 1984]
- reduction from previous problem
- at white node: black node:
- must connect added nodes to nearest neighbors
⇒ remainder is Ham. path
**Δ & hex grid graphs:** [Arkin, Fekete, Islam, Meijer, Mitchell, Núñez-Rodríguez, Polishchuk, Rappaport, Xiao 2009]

- solid \( \Rightarrow \) \( \Delta \) & \( \Box \) polynomial
- super-thin \( \Rightarrow \) \( \Box \) & \( \bigcirc \) polynomial, \( \Delta \) NP-hard
  \( \Rightarrow \) all faces are holes/outside
- thin \( \Delta \) & \( \Box \) NP-hard
  \( \Rightarrow \) every vertex on boundary
- max-deg. 4 \( \Delta \) NP-hard
- max-deg. 3 \( \Box \) & \( \bigcirc \) & \( \Delta \) NP-hard

\( \Rightarrow \Delta \) grid conversion:

- polygonal: \( \Delta \) polynomial, \( \bigcirc \) NP-hard
  \( \Rightarrow \) no holes/outside face share a vertex (antisuperthin)

**Settlers of Catan:** [Klaus Teuber 1995]

- hex grid
- longest road \( \Rightarrow 2 \) Victory Points
- mate-in-1 is NP-hard [Demaine, van Eyck, McKay 2011]
  - opponents serve as obstacles
  - enough resources to buy all roads
  - get longest road \( \Leftrightarrow \) Hamiltonian
- "mate-in-0" is NP-hard [Demaine, van Eyck, McKay 2011]
  - have longest road \( \Leftrightarrow \) Hamiltonian
Slitherlink [Nikoli 1989]
- given grid of squares each blank or 0–4
- goal: find cycle on grid lines such that numbered squares have that many incident edges
- reduction from Planar Ham. cycle [Yato - IPSJ 2000]
  - optional vertex gadget
  - required vertex gadget
  - (non)edge connections
- reduction from Hamiltonicity in grid graphs

Hashiwokakero: [Nikoli 1990] "build bridges"
- given nodes with desired degrees
- goal: build orthogonal (multi)edges to connect nodes & satisfy degrees
- reduction from Hamiltonicity in grid graphs [Andersson - IPL 2009]
  - 1s for boundary
  - internal node = 2 + # boundaries
Milling: (NC milling) [Arkin, Fekete, Mitchell - CGTA 2000]
- cut given region with given tool using shortest path staying inside region
- NP-hard for grid polygon & unit □ tool
- reduction from Hamiltonicity in grid graphs: Minkowski sum of vertices with unit □
  \[ \Rightarrow \text{set of all pairwise sums} \]

Lawn mowing: (laser/waterjet/sign cutting)
- path can go outside region
- NP-hard for grid polygon & unit □ tool
- same reduction: hurts length to leave region
- can even remove holes [Arkin, Fekete, Mitchell 2000]

3D printing: each layer is lawn mowing
Unit orthogonal segment intersection graphs:
- includes all grid graphs (rotated 45°)
⇒ Hamiltonicity NP-complete
[Arkin, Bender, Demaine, Fekete, Mitchell, Sethia - SICOMP 2005]

Minimum-turn milling: [Arkin et al. 2005]
- motivation: need to slow down for turns
- reduction from previous problem
  - segment → superthin rectangle
    → Minkowski sum with unit □
- need 4 turns per segment
  + 1 turn per transition
- 5n turns achievable ⇔ Hamiltonian