Recall: packing of $n$ squares without rotation into a square is strongly NP-complete \[L2\]

Edge-unfolding polyhedra: given a polyhedron, cut along edges to unfold flat without overlap

- not always possible \[Biedl et al. & Bern et al. 1998\]
- strongly NP-hard \[Abel & Demaine 2011\]
even for orthogonal polyhedra topologically sphere

Reduction from Square Packing:
- infrastructure: polyhedron with square with tower with squares & "atoms" on side
- "pipe" is super long but can move out
  $\Rightarrow$ squares must pack inside base of tower
- atoms are universal: can turn left/right/straight in 2D unfolding & left/right/straight on tower surface
  $\Rightarrow$ can connect & place squares as in any (slightly perturbed) packing, then exit via pipe
- lots of details e.g. shrink squares slightly to enable perturbation
Snake cube puzzle: AKA Cubra circa 1990
- given chain of unit cubes each with specified “turn angle” of 0 or 90° (elastic through centers)
- goal: fold it into larger cube (exactly)
- NP-hard
[Abel, Demaine, Demaine, Eisenstat, Lynch, Schardl 2012]

Reduction from 3-Partition:
- infrastructure:
  - fill cube to leave \( x \times y \times z \) box
  - fill box to leave “hub & slots” shape
  - each hub is \( 8 \times t \times \text{huge} \)
- \( a_i \) gadget: \( 8a_i \) must go in 1 hub
  - 8 to avoid coming back to same \( 4 \times 4 \times 4 \) voxel
  - connected together by zig-zag gadget
  - zig-zag is universal:
    - \( 2 \times 2 \times 2 \) can turn/go straight
      \( \Rightarrow \) fill Hamiltonian shapes scaled \( 2 \times \)
    - \( 2 \times 2 \times 2 \) refinement makes any shape Hamiltonian
      \( \Rightarrow \) \( 4 \times 4 \times 4 \) refinement makes fillable by zig-zag
  - parity issue: snake alternates in cell parity
  - claim: can start & end at any faces of opposite parity
Disk packing: pack n given disks into given shape

- motivation: computational origami design (tree method - see Lang)
- strongly NP-hard [Demaine, Fekete, Lang-OSME 2010]

Reduction from 3-Partition:

- infrastructure:
  - build \( n/3 \) symmetric \( \delta \) pockets
  - equilateral \( \Delta \): forced packing
  - square target: forced packing
  + repeated subdivision with forced packings
  + fill all other pockets by repeatedly adding maximal disks, until small enough (depth \( \approx \log n \))

- triple gadget: (in symmetric pocket)
  - scale \( a_i \)'s & \( t \) so that \( t = 1 \)
  - shrink center disk by \( -1/N \)
  - shrink \( a_i \) disk by \( -1/N^2 \), \( \bigcup_{\bigcup_{\text{big}}} \)
  - grow it by \( +a_i/N \)

- key property: disks fit \( \Leftrightarrow a_i + a_j + a_k \leq t \) (proof by geometry + Taylor series)
Clickomania: [Schuessler ~2000?]
- given rectangular grid of colored squares
- move = remove connected group of >1 square of the same color
- remaining squares fall within each column
- empty columns disappear

- polynomial for one row or column
  - reduces to CFG parsing
- NP-hard for
  - 2 columns & 5 colors
  - 5 columns & 3 colors
- OPEN: 2 rows?
  - 2 colors?

Reduction from 3-Partition:
- left column mostly checkerboard except middle & interspersed red $\Box$s to measure $t$’s
  - collapses $\Leftrightarrow$ red $\Box$s removed
- right column has $a_i$ groups $\leftarrow$ scaled by $B = \frac{4}{3}n$
  - red squares on top
- details: spacing out groups & reds while still getting alignment

[Biedl, Demaine, Demaine, Fleischer, Jacobsen, Munro 2000]

necessary: encoding in unary
Tetris: [Alexey Pazhitnov 1985]
- rectangular board
- tetromino blocks come one at a time
  - 4 unit squares joined edge-to-edge
- can rotate block as it falls from sky
- filled lines disappear
- stack to sky \( \Rightarrow \) die

- perfect information version:
  - know entire sequence of pieces to come
  - initial board position given
- \( \text{NP-complete to} \) [Breukelaar, Demaine, Hohenberger, Hoogeboom, Kosters, Liben-Nowell 2003]
  - survive
  - approximate \# lines/Tetrises/time until death up to a factor of \( n^{1-\varepsilon} \)

Reduction from 3-Partition: \( \Rightarrow \) necessary: encoding in unary
- initial board = \( \frac{n}{3} \) buckets of “depth” \( \varepsilon \)
- \( a_i \) encoded as \( \square \), \( \square \), \( \square \), \( \square \), \( \square \)
  - claim: entire gadget must go in one bucket
- finale = \( \square \) \( \frac{n}{3} \), \( \square \), \( \square \) \( \frac{n}{3} \varepsilon + 4 \)
- initially empty board
- $O(1)$ rows or columns
- restricted piece sets (e.g. \[ \begin{array}{c} \hline \\ \end{array} \])
- no last-minute slides
- 2-player: PSPACE-complete?
- online Tetris?

1-planarity: draw a given graph in the plane such that each edge crosses $\leq 1$ other

NP-complete [Grigoriev & Bodlaender - Alg. 2007]

**Reduction from 3-Partition:**
- **uncrossable edge gadget:**
  - (denoted by thick edge)
- **double wheel gadget:**
  - unique embedding
  - one for $A$
  - one for triples
  - separate triples with thick edges every $t$ hours around triples gadget
- $a_i$-gadget:
  - $A$ center
  - triples center
GeoLoop & Ivan's Hinge puzzles: piano-hinged dissection
\[ \Rightarrow \text{NP-complete from 3-Partition} \]

**Ruler folding:**
- given carpenter's ruler with lengths \(a_1, a_2, \ldots, a_n\)
- goal: fold to fit in 1D box of length \(L\)

- weakly NP-complete [Hopcroft, Joseph, Whitesides-1985]
- pseudopolynomial (like 2-Partition)

**Reduction from \((2-)*\)Partition:**
- idea: Partition solvable \(\iff\) can assign signs to \(a_i\)'s such that \(\sum_{i} \pm a_i = 0\)
- folding flips sign; unfolding leaves sign
\[ \Rightarrow \text{can fold ends together} \iff \text{Partition solvable} \]
- construction: \( 2B, B, a_1, a_2, \ldots, a_n, B, 2B \)
\[ \Rightarrow \sum_{i} a_i \]
\[ \Rightarrow 2B's \text{ will be aligned & fit inside length-2B box} \]
\[ \Rightarrow \text{can fold ends together} \iff \text{Partition solvable} \]
Map folding (simple): given crease pattern, can it fold flat by sequence of simple folds?

- weakly NP-hard \cite{Arkin, Bender, Demaine, Demaine, Mitchell, Sethia, Skiena - 2000}

for orthogonal paper & orthogonal creases or square paper & 45° orthog. creases

Reduction from Partition:
- Similar to Ruler Folding
- 2 vertical creases check y extent against frame
- horizontal creases done before or after check
  \[ \text{if ruler folded} \quad \text{5 if not} \]
- force square paper into orthogonal shape:

\[ \text{OPEN: strongly NP-hard? pseudopolynomial?} \]