Problem Set 8

This problem set is due Wednesday, November 9 at noon.

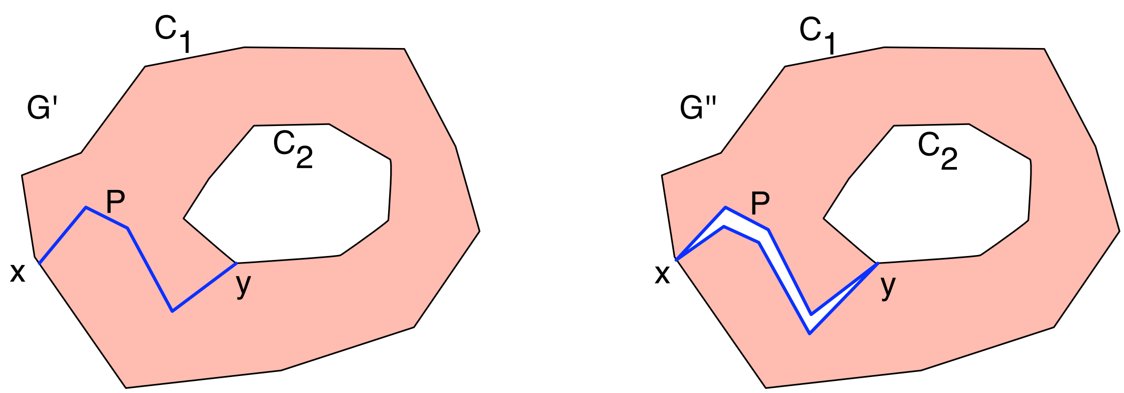
1. Recall that in lecture 14 we represented the edges of the dense distance graph in a matrix A_i. We saw that performing a single iteration of Belamn-Ford amounts to finding all column minima of A_i, and showed that A_i can be partitioned into square Monge submatrices and that the column minima of a m-by-n Monge matrix can be found in $O(m + n)$ time using the SMAWK algorithm.

   In the case we discussed in class, the nodes of the dense distance graph were the nodes of a single simple cycle C, and the length of an edge of the dense distance graph for G_i corresponded to the length of the shortest path in G_i between the corresponding nodes of C.

   In this problem we consider the case where the nodes of the dense distance graph lie on two simple cycles instead of just one. This case arises when using an r–decomposition instead of a single cycle to compute shortest paths with negative lengths (this leads to improved running time of $O(n \log^2 n / \log \log n)$), as well as in the generalization of the shortest path algorithm to higher genus.

   (a) Let C_1, C_2 be two simple cycles in G. Assume, without loss of generality that C_1 is the infinite face of G. Let G' be the subgraph of G not enclosed by C_2. Let A' be the matrix whose rows correspond to the nodes of C_1 and whose columns correspond to the nodes of C_2, where $A'_{i,j}$ is the length of a shortest path in G' from node $i \in C_1$ to node $j \in C_2$. Is there an ordering of the nodes of C_1 and C_2 such that A' is Monge? (As we saw in class, the direction of the Monge inequality is not important)

   (b) Consider a simple x-to-y path P in G' where $x \in C_1$ and $y \in C_2$. Recall the concept of cutting a graph open along a path from Lecture 15. Let G'' be the graph obtained by cutting G' open along P. By this we mean make two copies of P, duplicating every edge of P and every internal node of P, but not x and y. This way the two copies of P are connected at x and y and bound a single face of G''. See figure.



   Let A'' be the matrix whose rows correspond to the nodes of C_1 and whose columns correspond to the nodes of C_2, where $A''_{i,j}$ is the length of a shortest path in G'' from node $i \in C_1$ to node $j \in C_2$. Is there an ordering of the nodes of C_1 and C_2 such that A'' is Monge? Again, the direction of the Monge inequality does not matter.

   (c) Argue that $\forall i \in C_1 \forall j \in C_2 : A'_{i,j} \le A''_{i,j}$. When does $A'_{i,j} = A''_{i,j}$?

   (d) Using the concepts above, describe how to find all column minima of A' in $O(|C_1| + |C_2|)$ time. Explain why your solution is correct (you do not have to provide detailed proofs). There is a hint available on the website (please indicate if you needed the hint or not. This will not affect your grade).