Problem Set 8
This problem set is due Wednesday, November 9 at noon.

1. Recall that in lecture 14 we represented the edges of the dense distance graph in a matrix $A_i$. We saw that performing a single iteration of Bellman-Ford amounts to finding all column minima of $A_i$, and showed that $A_i$ can be partitioned into square Monge submatrices and that the column minima of a $m$-by-$n$ Monge matrix can be found in $O(m + n)$ time using the SMAWK algorithm.

In the case we discussed in class, the nodes of the dense distance graph were the nodes of a single simple cycle $C$, and the length of an edge of the dense distance graph for $G_i$ corresponded to the length of the shortest path in $G_i$ between the corresponding nodes of $C$.

In this problem we consider the case where the nodes of the dense distance graph lie on two simple cycles instead of just one. This case arises when using an $r$–decomposition instead of a single cycle to compute shortest paths with negative lengths (this leads to improved running time of $O(n \log^2 n / \log \log n)$), as well as in the generalization of the shortest path algorithm to higher genus.

(a) Let $C_1, C_2$ be two simple cycles in $G$. Assume, without loss of generality that $C_1$ is the infinite face of $G$. Let $G'$ be the subgraph of $G$ not enclosed by $C_2$. Let $A'$ be the matrix whose rows correspond to the nodes of $C_1$ and whose columns correspond to the nodes of $C_2$, where $A'_{i,j}$ is the length of a shortest path in $G'$ from node $i \in C_1$ to node $j \in C_2$. Is there an ordering of the nodes of $C_1$ and $C_2$ such that $A'$ is Monge? (As we saw in class, the direction of the Monge inequality does not matter)

(b) Consider a simple $x$-to-$y$ path $P$ in $G'$ where $x \in C_1$ and $y \in C_2$. Recall the concept of cutting a graph open along a path from Lecture 15. Let $G''$ be the graph obtained by cutting $G'$ open along $P$. By this we mean make two copies of $P$, duplicating every edge of $P$ and every internal node of $P$, but not $x$ and $y$. This way the two copies of $P$ are connected at $x$ and $y$ and bound a single face of $G''$. See figure.

Let $A''$ be the matrix whose rows correspond to the nodes of $C_1$ and whose columns correspond to the nodes of $C_2$, where $A''_{i,j}$ is the length of a shortest path in $G''$ from node $i \in C_1$ to node $j \in C_2$. Is there an ordering of the nodes of $C_1$ and $C_2$ such that $A''$ is Monge? Again, the direction of the Monge inequality does not matter.

(c) Argue that $\forall i \in C_1 \forall j \in C_2 : A'_{i,j} \leq A''_{i,j}$. When does $A'_{i,j} = A''_{i,j}$?

(d) Using the concepts above, describe how to find all column minima of $A'$ in $O(|C_1| + |C_2|)$ time. Explain why your solution is correct (you do not have to provide detailed proofs). There is a hint available on the website (please indicate if you needed the hint or not. This will not affect your grade).