# Problem Set 6

This problem set is due Wednesday, October 26 at noon.

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For this problem set it is important to know that the separation property is defined somewhat differently for contraction-bidimensional problems. Let $(A, B, S)$ be a separation of $G$ and $Z \subseteq V(G)$ be an optimal solution to a contraction bidimensional problem $\Pi$ in $G$. Let $G_A$ denote the graph obtained by *contracting* each connected component of $G[B]$ into its adjacent vertex of $S$ with smallest index, and define $G_B$ similarly. Let $Z_A$ denote an optimal solution to $\Pi$ in $G_A$ and $Z_B$ and optimal solution in $G_B$. We say $\Pi$ has the separation property if

$$|Z_A| \le |Z - B| + O(|S|) \text{ and } |Z_B| \le |Z - A| + O(|S|) \ .$$

1. Show that the minimum connected dominating set problem admits a PTAS in apex-minor-free graphs.

   A dominating set in a graph $G$ is a set $D \subseteq V(G)$ such that $D \cup N(D) = V(G)$, where $N(D)$ is the set of all vertices that are neighbors of some vertex of $D$. It is called a connected dominating set if $G[D]$ is connected.

   **Solution:** First, note that upon contracting an edge, a connected dominating set (CDS) remains connected and dominating, and hence the size of a minimum CDS does not increase when contracting edges. Furthermore, the size of a CDS on the graph $\Gamma_k$ is $\Omega(k^2)$. Therefore, CDS is contraction-bidimensional.

   Next, we show that CDS has the separation property. Let $(A, B, S)$ be a separation in $G$, and $G_A$, $G_B$, $Z_A$, and $Z_B$ defined as above. Note that when obtaining $G_A$ from $G$, we are only contracting edges. Initialize a set $D_A$ by $Z \cap A$. Upon contracting a connected component $C$ of $G[B]$ into a vertex $v_C \in S$, add $v_C$ to $D_A$ if and only if $v_C \in Z$ or $C \cap Z \neq \emptyset$. This way, $D_A$ is a CDS of $G_A$ that we obtain by keeping track of $Z$ as we are contracting $G[B]$. Furthermore, $|D_A| \le |Z - B| + |S|$. Since $Z_A$ is an optimal CDS in $G_A$, we have $|Z_A| \le |D_A|$, and so CDS fulfills the separation property.

   In order to obtain a PTAS for CDS in apex-minor-free graphs, we proceed by the framework introduced in the lecture. Since CDS is contraction-bidimensional with the separation property, it follows that a core exists and can be shrunk so as to obtain a set $X'$ of size at most $\frac{\epsilon}{3}\mathrm{OPT}$, so that $\mathrm{tw}(G - X')$ is a constant. Let $N(X')$ denote the neighbors of $X'$ in $G$, excluding $X'$. Using dynamic programming on graphs of bounded treewidth, solve the following problem on $G - X'$: find a vertex set $D'$ of minimum size such that every vertex of $G - X' - N(X')$ is dominated and every connected component of $G[D']$ has a vertex in $N(X')$. Note that for an optimal CDS $D^\star$ of $G$, $D^\star - X'$ has this property in $G - X'$, and so $|D'| \le |D^\star|$. Now $D' \cup X'$ is a dominating set of $G$

that consists of at most $|X'|$ connected components. This can be augmented to a CDS $D$ of $G$ by adding at most $2(|X'|-1)$ vertices as shown below. Hence

$$|D| \le |D^\star| + 3|X'| \le (1+\epsilon)\mathrm{OPT}\ .$$

(This shows that CDS is reducible.)

To see how to obtain a CDS from a dominating set with few components, observe the following. Let $C$ be a connected component of a dominating set and let $C'$ be another connected component that is closest to $C$. The shortest path between $C$ and $C'$ cannot contain more than 2 internal vertices since all internal vertices are dominated and hence adjacent to or included in some connected component of the dominating set.

2. Show that the connected vertex cover problem admits a PTAS in $H$-minor-free graphs.

   Recall that a vertex cover in a graph is a set of vertices $Z$ such that every edge of the graph has at least one endpoint in $Z$; it is called a connected vertex cover if $G[Z]$ is connected.

   **Solution:** Even though connected vertex cover (CVC) is contraction bidimensional, this would only give us a PTAS on apex-minor-free graphs. In order to obtain a PTAS on $H$-minor-free graphs, we show directly how to find a core for this problem. To this end, consider a maximal matching $M$ in $G$ (not necessarily maximum). Let $X := V(M)$, i.e. both endpoints of all the edges of $M$. Since any vertex cover must contain at least one endpoint of each edge of $M$, we have that the size of $|X|$ is at most twice that of the minimum vertex cover, and hence at most twice that of a minimum CVC. Also, $G-X$ is an independent set, and so $\mathrm{tw}(G-X)=0$.

   Recall that the shrinking of the core does not depend on bidimensionality and can be done on any given set $X$ in $H$-minor-free graphs. So, we obtain a set $X'$ with $|X'| \le \frac{\epsilon}{2}\mathrm{OPT}$ such that $G-X'$ has bounded treewidth. It remains to show that CVC is reducible. The proof is analogous to the one for CDS. Compute a set $Z'$ of minimum size in $G-X'$ such that $Z'$ covers every edge in $G-X'$ and every connected component of $G[Z']$ has a vertex in $N(X')$. Now $Z' \cup X'$ is a vertex cover of $G$ that consists of at most $|X'|$ connected components and can be turned into a CVC by adding at most $|X'|-1$ vertices.