Problem Set 5

This problem set is due Wednesday, October 19 at noon.

1. For $m > 2k$, let G be an $m \times m$ grid, and G' be the central $(m-2k) \times (m-2k)$ subgrid of G. Let N be a set of at least k^4 vertices in G'.

   (a) Show that one can designate a side of G as the x-axis and the other one as the y-axis in such a way that at least k^2 vertices of N have different y-coordinates.

   (b) Let $N' = \{v_1, \ldots, v_{k^2}\} \subseteq N$ be a set of exactly k^2 vertices with different y-coordinates and assume that the v_j are sorted by increasing y-coordinate. For $0 \leq i < k$, let $N_i = \{v_j : ki \leq j < k(i+1)\}$. Show (essentially by picture) that there exist k disjoint paths in G such that each path contains exactly one vertex out of each of $N_0, \ldots, N_{k-1}$.

   (c) Show that G contains a model of a $k \times k$-grid K in which the image of each vertex of K in G contains exactly one vertex of N'.

2. Let H be an apex graph with $|H| \geq 2$, $k = 14|V(H)| - 22$, and $m > 2k$.

   (a) Let $\mathcal{M}$ be an $m \times m$-grid augmented by some additional edges but no additional vertices and that excludes H as a minor. Show that every vertex of $\mathcal{M}$ may be adjacent to at most k^4 vertices in the central $(m-2k) \times (m-2k)$-subgrid of $\mathcal{M}$.

   (b) Let G be a graph that excludes H as a minor and has treewidth at least $m^{4|V(H)|^2(m+2)}$. Show that G can be contracted to a graph R with the following properties: R is an $(m-2k) \times (m-2k)$-grid with additional edges but no additonal vertices; furthermore, every vertex of R is adjacent to at most $(k+1)^6$ inner vertices of the grid. An inner vertex is a vertex that is not on the boundary of the grid.

   For this problem, you may use the following theorems without proof:

   (i) Every planar graph H is a minor of an $r \times r$-grid for $r = 14|V(H)| - 24$.

   (ii) Every graph of treewidth at least $m^{4r^2(m+2)}$ contains either K_r or an $m \times m$-grid as a minor.

3. Show that for any apex graph H, the class of H-minor-free graphs has bounded local treewidth. To this end, use the previous problem and consider the maximum number of vertices that one can reach from any vertex in R in i steps; then relate it to the fact that after at most d steps (where d is the diameter), one should have reached every vertex.

   Note that as we saw in the lecture, this is actually an equivalence: The minor-closed classes of graphs that have bounded local treewidth are exactly the apex-minor-free classes. Also, these classes actually have linear local treewidth - even though our proof here shows only an exponential bound.